Chapter 15

Partial Differentiation

A function of several variables can be differentiated with respect to one variable at a time.

The rate of change of a function of several variables is not just a single function, since the independent variables may vary in different ways. All the rates of change, for a function of n variables, are described by n functions called its *partial derivatives*. This chapter begins with the definition and basic properties of partial derivatives. Methods for computation, including the chain rule, are presented along with a geometric interpretation in terms of tangent planes. The next chapter continues the development with topics including implicit differentiation, gradients, and maxima and minima.

15.1 Introduction to Partial Derivatives

The partial derivatives of a function of several variables are its ordinary derivatives with respect to each variable separately.

In this section we define partial derivatives and practice computing them. The geometric significance of partial derivatives and their use in computing tangent planes are explained in the next section.

Consider a function $f(x, y)$ of two variables. If we treat y as a constant, f may be differentiated with respect to x. The result is called the *partial derivative of f with respect to x* and is denoted by f_x. If we let $z = f(x, y)$, we write

$$f_x = \frac{\partial z}{\partial x}.$$

These symbols[1] are analogous to those we used in one-variable calculus: $f'(x) = dy/dx$.

The partial derivative with respect to y is similarly defined by treating x as a constant and differentiating $f(x, y)$ with respect to y.

[1] The symbol ∂ seems to have first been used by Clairaut and Euler around 1740 to avoid confusion with d. The notation $D_x f$ or $D_1 f$ for f_x is also used.

Example 1 (a) If $f(x, y) = xy + e^x \cos y$, compute f_x and f_y.
(b) For f as in (a), calculate $f_x(1, \pi/2)$.
(c) If $z = x^2y^3 + x^3y^4 - e^{xy^2}$, calculate $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution (a) Treating y as a constant and differentiating with respect to x, we get

$$f_x(x, y) = y + e^x \cos y.$$

Differentiating with respect to y and considering x as a constant gives

$$f_y(x, y) = x - e^x \sin y.$$

(b) Substituting $x = 1$ and $y = \pi/2$, we get

$$f_x(1, \pi/2) = \pi/2 + e^1 \cos(\pi/2) = \pi/2.$$

(c) Here we again hold y constant and calculate the x derivative:

$$\frac{\partial z}{\partial x} = 2xy^3 + 3x^2y^4 - y^2e^{xy^2}.$$

Similarly,

$$\frac{\partial z}{\partial y} = 3x^2y^2 + 4x^3y^3 - 2yxe^{xy^2}. \ \blacktriangle$$

In terms of limits, partial derivatives are given by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

See Fig. 15.1.1.

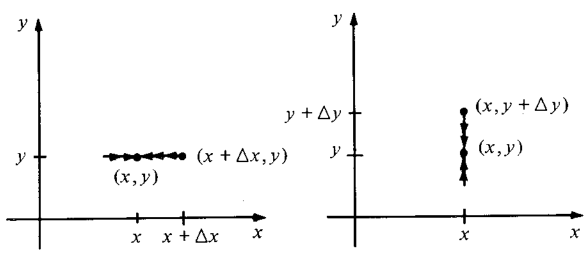

Figure 15.1.1. The partial derivatives f_x and f_y are limits of difference quotients along the horizontal and vertical paths shown here.

Partial derivatives of functions $f(x, y, z)$ of three variables are defined similarly. Two variables are treated as constant while we differentiate with respect to the third.

Example 2 (a) Let $f(x, y, z) = \sin(xy/z)$. Calculate $f_z(x, y, z)$ and $f_z(1, 2, 3)$.
(b) Evaluate

$$\frac{\partial}{\partial y} \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

at $(0, 1, 1)$.
(c) Write the result in (b) as a limit.

Solution (a) We differentiate $\sin(xy/z)$ with respect to z, thinking of x and y as constants; the result is

$$f_z(x, y, z) = \cos(xy/z)(-xy/z^2) = -(xy/z^2)\cos(xy/z).$$

Substituting $(1,2,3)$ for (x, y, z) gives

$$f_z(1,2,3) = -\frac{1\cdot 2}{3^2}\cos\left(\frac{1\cdot 2}{3}\right) = -\frac{2}{9}\cos\left(\frac{2}{3}\right).$$

(b) Treating x and z as constants, and using the chain rule of one-variable calculus,

$$\begin{aligned}\frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}} &= \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}\\ &= \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}\cdot 2y\\ &= \frac{-y}{(x^2+y^2+z^2)^{3/2}}.\end{aligned}$$

At $(0,1,1)$ this becomes

$$\frac{-1}{(0^2+1^2+1^2)^{3/2}} = \frac{-1}{2\sqrt{2}}.$$

(c) In general,

$$\lim_{\Delta y\to 0}\frac{f(x, y+\Delta y, z)-f(x, y, z)}{\Delta y} = f_y(x, y, z).$$

In case (b), this becomes

$$\lim_{\Delta y\to 0}\frac{1}{\Delta y}\left[\frac{1}{\sqrt{(1+\Delta y)^2+1}}-\frac{1}{\sqrt{2}}\right] = -\frac{1}{2\sqrt{2}}. \blacktriangle$$

Partial Differentiation

If f is a function of several variables, to calculate the partial derivative with respect to a certain variable, treat the remaining variables as constants and differentiate as usual by using the rules of one-variable calculus.

If $z = f(x, y)$ is a function of two variables, the partial derivatives are denoted $f_x = \partial z/\partial x$ and $f_y = \partial z/\partial y$.

If $u = f(x, y, z)$ is a function of three variables, the partial derivatives are denoted $f_x = \partial u/\partial x$, $f_y = \partial u/\partial y$, and $f_z = \partial u/\partial z$.

As in one-variable calculus, the letters for the variables do not always have to be x, y, z.

Example 3 If $h = rs^2\sin(r^2+s^2)$, find $\partial h/\partial s$.

Solution Holding r constant, we get

$$\begin{aligned}\frac{\partial h}{\partial s} &= 2rs\sin(r^2+s^2) + rs^2\cdot 2s\cdot\cos(r^2+s^2)\\ &= 2rs\left[\sin(r^2+s^2) + s^2\cos(r^2+s^2)\right]. \blacktriangle\end{aligned}$$

Partial derivatives may be interpreted in terms of rates of change, just as derivatives of functions of one variable.

Example 4 The temperature (in degrees Celsius) near Dawson Creek at noon on April 15, 1901 is given by $T = -(0.0003)x^2y + (0.9307)y$, where x and y are the latitude and longitude (in degrees). At what rate is the temperature changing if we proceed directly north? (The latitude and longitude of Dawson Creek are $x = 55.7°$ and $y = 120.2°$.)

Solution Proceeding directly north means increasing the latitude x. Thus we calculate

$$\frac{\partial T}{\partial x} = -(0.0003)\cdot 2xy = -(0.0003)\cdot 2\cdot(55.7)\cdot(120.2) \approx -4.017.$$

So the temperature drops as we proceed north from Dawson Creek, at the instantaneous rate of 4.017°C per degree of latitude. ▲

Since the partial derivatives are themselves functions, we can take their partial derivatives to obtain higher derivatives. For a function of two variables, there are four ways to take a second derivative. If $z = f(x, y)$, we may compute

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}, \qquad f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2},$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x}, \qquad f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y}.$$

Example 5 Compute the second partial derivatives of $z = xy^2 + ye^{-x} + \sin(x-y)$.

Solution We compute the first partials:

$$\frac{\partial z}{\partial x} = y^2 - ye^{-x} + \cos(x-y)$$

and

$$\frac{\partial z}{\partial y} = 2xy + e^{-x} - \cos(x-y).$$

Now we differentiate again:

$$\frac{\partial^2 z}{\partial x^2} = ye^{-x} - \sin(x-y), \qquad \frac{\partial^2 z}{\partial y^2} = 2x - \sin(x-y),$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2y - e^{-x} + \sin(x-y),$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2y - e^{-x} + \sin(x-y). \text{ ▲}$$

Example 6 (a) If $u = y\cos(xz) + x\sin(yz)$, calculate $\partial^2 u/\partial x\,\partial z$ and $\partial^2 u/\partial z\,\partial x$. (b) Let $f(x,y,z) = e^{xy} + z\cos x$. Find f_{zx} and f_{xz}.

Solution (a) We find $\partial u/\partial x = -yz\sin(xz) + \sin(yz)$ and $\partial u/\partial z = -xy\sin(xz) + xy\cos(yz)$. Thus

$$\frac{\partial^2 u}{\partial z\,\partial x} = -y\sin(xz) - xyz\cos(xz) + y\cos(yz).$$

Differentiation of $\partial u/\partial z$ with respect to x yields

$$\frac{\partial^2 u}{\partial x\,\partial z} = -y\sin(xz) - xyz\cos(xz) + y\cos(yz).$$

(b) $f_x(x,y,z) = ye^{xy} - z\sin x$; $f_z(x,y,z) = \cos x$; $f_{zx}(x,y,z) = (\partial/\partial x)(\cos x) = -\sin x$; $f_{xz}(x,y,z) = (\partial/\partial z)(ye^{xy} - z\sin x) = -\sin x$. ▲

In the preceding examples, note that the mixed partials taken in different orders, like $\partial^2 z/\partial x\,\partial y$ and $\partial^2 z/\partial y\,\partial x$, or f_{xz} and f_{zx}, are equal. This is no accident.

Theorem: Equality of Mixed Partial Derivatives

If $u = f(x, y)$ has continuous second partial derivatives, then the mixed partial derivatives are equal; that is,

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \quad \text{or} \quad f_{yx} = f_{xy}\,.$$

Similar equalities hold for mixed partial derivatives of functions of three variables.

L. Euler discovered this result around 1734 in connection with problems in hydrodynamics. To prove it requires the notions of continuity and limit for functions of two variables.[2]

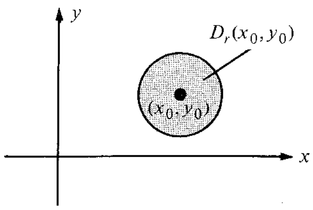

Figure 15.1.2. The disk $D_r(x_0, y_0)$ consists of the shaded region (excluding the solid circle).

Let us write $d((x, y), (x_0, y_0)) = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ for the distance between (x, y) and (x_0, y_0), with a similar notation $d((x, y, z), (x_0, y_0, z_0))$ in space. The *disk* $D_r(x_0, y_0)$ of radius r centered at (x_0, y_0) is, by definition, the set of all (x, y) such that $d((x, y), (x_0, y_0)) < r$, as shown in Fig. 15.1.2. The limit concept now can be defined by the same ε, δ technique as in one variable calculus.

The ε, δ Definition of Limit

Suppose that f is defined on a region which includes a disk about (x_0, y_0), but need not include (x_0, y_0) itself. We write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$$

if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x, y) - l| < \varepsilon$ whenever $0 < d((x, y), (x_0, y_0)) < \delta$. A similar definition is made for functions of three variables.

The ε, δ definition of limit may be rephrased as follows: for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x, y) - l| < \varepsilon$ if (x, y) lies in $D_\delta(x_0, y_0)$.

The similarity between this definition and the one in Chapter 11 should be evident. The rules for limits, including rules for sums, products, and quotients, are analogous to those for functions of one variable.

Example 7 Prove the "obvious" limit, $\lim_{(x,y)\to(x_0,\,y_0)} x = x_0$, using ε's and δ's.

Solution Let $\varepsilon > 0$ be given and let $f(x, y) = x$ and $l = x_0$. We seek a number $\delta > 0$ such that $|f(x, y) - l| < \varepsilon$ whenever $d((x, y), (x_0, y_0)) < \delta$, that is, such that $|x - x_0| < \varepsilon$ whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$. However, note that

$$|x - x_0| = \sqrt{(x - x_0)^2} \leqslant \sqrt{(x - x_0)^2 + (y - y_0)^2}\,,$$

so if we choose $\delta = \varepsilon$, $d((x, y), (x_0, y_0)) < \delta$ will imply $|x - x_0| < \varepsilon$. ▲

[2] If you are not interested in the theory of calculus, you may skip to p. 772. Consult your instructor.

Example 8 Find $\lim_{(x,y)\to(0,0)} \dfrac{x^3 + 2x^2 + xy^2 + 2y^2}{x^2 + y^2}$.

Solution The numerator and denominator vanish when $(x, y) = (0,0)$. The numerator factors as $(x^2 + y^2)(x + 2)$, so we may use the replacement rule and algebraic rules to get

$$\lim_{(x,y)\to(0,0)} \frac{(x^2 + y^2)(x + 2)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} (x + 2) = 0 + 2 = 2. \blacktriangle$$

Example 9 Show that

$$\lim_{(x,y)\to(0,0)} \frac{\partial}{\partial x}\sqrt{x^2 + y^2}$$

does not exist. [*Hint*: Look at the limits along the x and y axes.]

Solution $(\partial/\partial x)\sqrt{x^2 + y^2} = x/\sqrt{x^2 + y^2}$ if $(x, y) \neq (0,0)$. Thus

$$\lim_{(x,y)\to(0,0)} \frac{\partial}{\partial x}\sqrt{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2 + y^2}}.$$

If we approach $(0,0)$ on the y axis—that is, along points $(0, y)$—we get zero. Thus the limit, if it exists, is zero. On the other hand, if we approach $(0,0)$ along the positive x axis, we have $y = 0$ and $x > 0$; then $x/\sqrt{x^2 + y^2} = 1$ because $x/\sqrt{x^2} = 1$, so the limit is 1. Since we obtain different answers in different directions, the actual limit cannot exist. $\blacktriangle$

We can base the concept of continuity on that of limits, just as we did in Section 1.2.

Definition of Continuity

Let f be defined in a disk about (x_0, y_0). Then we say f is *continuous* at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

There is a similar definition for functions of three variables.

Most "reasonable" functions of several variables are continuous, although this may not be simple to prove from the definition. Here is an example of how to do this.

Example 10 (a) If $f(x)$ and $g(y)$ are continuous functions of x and y, respectively, show that $h(x, y) = f(x)g(y)$ is continuous. (b) Use (a) to show that $e^x \cos y$ is continuous.

Solution (a) We must show that for any (x_0, y_0), $\lim_{(x,y)\to(x_0,y_0)} f(x)g(y) = f(x_0)g(y_0)$. To this end, we manipulate the difference:

$$\begin{aligned}&|f(x)g(y) - f(x_0)g(y_0)| \\ &\quad = |f(x)g(y) - f(x_0)g(y) + f(x_0)g(y) - f(x_0)g(y_0)|\end{aligned}$$

$$
\begin{aligned}
&\leqslant |(f(x)-f(x_0))g(y)| + |f(x_0)(g(y)-g(y_0))| \\
&= |f(x)-f(x_0)|\,|g(y)| + |f(x_0)|\,|g(y)-g(y_0)| \\
&\leqslant |f(x)-f(x_0)|(|g(y_0)| + |g(y)-g(y_0)|) + |f(x_0)|\,|g(y)-g(y_0)| \\
&= |f(x)-f(x_0)|\,|g(y_0)| + |f(x_0)|\,|g(y)-g(y_0)| \\
&\quad + |f(x)-f(x_0)|\,|g(y)-g(y_0)|.
\end{aligned}
$$

Now let $\varepsilon > 0$ be given. We may choose $\varepsilon_1 > 0$ so small that we have the inequality $\varepsilon_1(|g(y_0)| + |f(y_0)|) + \varepsilon_1^2 < \varepsilon$, by letting ε_1 be the smaller of

$$\frac{\varepsilon}{|g(y_0)| + |f(y_0)| + 1} \quad \text{and} \quad 1.$$

Since f and g are continuous, there exists $\delta_1 > 0$ such that, when $|x - x_0| < \delta_1$, $|f(x) - f(x_0)| < \varepsilon_1$, and there exists δ_2 such that when $|y - y_0| < \delta_2$, we have the inequality $|g(x) - g(x_0)| < \varepsilon_1$. Let δ be the smaller of δ_1 and δ_2.

Now if $d((x, y), (x_0, y_0)) < \delta$, we have $|x - x_0| < \delta \leqslant \delta_1$ and $|y - y_0| < \delta \leqslant \delta_2$. so

$$
\begin{aligned}
|f(x)g(y) - f(x_0)g(y_0)| &\leqslant |f(x)-f(x_0)|\,|g(y_0)| + |f(x_0)|\,|g(y)-g(y_0)| \\
&\quad + |f(x)-f(x_0)|\,|g(y)-g(y_0)| \\
&\leqslant \varepsilon_1|g(y_0)| + |f(x_0)|\varepsilon_1 + \varepsilon_1 \cdot \varepsilon_1 < \varepsilon.
\end{aligned}
$$

Thus we have proven that $\lim_{(x,y)\to(x_0,y_0)} = f(x_0)g(y_0)$, so $f(x)g(y)$ is continuous.

(b) We know from one-variable calculus that the functions $f(x) = e^x$ and $g(y) = \cos y$ are differentiable and hence continuous, so by part (a), $f(x)g(y)$ is continuous. ▲

Using the ideas of limit and continuity, we can now give the proof of the equality of mixed partials; it uses the mean value theorem for functions of one variable.

Proof of the equality of mixed partial derivatives

Consider the expression

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) - f(x_0, y_0 + \Delta y) + f(x_0, y_0). \tag{1}$$

We fix y_0 and Δy and introduce the function

$$g(x) = f(x, y_0 + \Delta y) - f(x, y_0),$$

so that the expression (1) equals $g(x_0 + \Delta x) - g(x_0)$. By the mean value theorem for functions of one variable, this equals $g'(\bar{x})\Delta x$ for some $\bar{x}$ between x_0 and $x_0 + \Delta x$. Hence (1) equals

$$\left[\frac{\partial z}{\partial x}(\bar{x}, y_0 + \Delta y) - \frac{\partial z}{\partial x}(\bar{x}, y_0)\right]\Delta x.$$

Applying the mean value theorem again, we get, for (1),

$$\frac{\partial^2 z}{\partial y\,\partial x}(\bar{x}, \bar{y})\Delta x\,\Delta y.$$

Since $\partial^2 z/\partial y\,\partial x$ is continuous and $(\bar{x}, \bar{y}) \to (x_0, y_0)$ as $(\Delta x, \Delta y) \to (0,0)$, it follows that

$$
\begin{aligned}
&\frac{\partial^2 z}{\partial y\,\partial x}(x_0, y_0) \\
&= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{[f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) - f(x_0, y_0 + \Delta y) + f(x_0, y_0)]}{\Delta x\,\Delta y}.
\end{aligned}
\tag{2}
$$

The right-hand side of formula (2) is symmetric in x and y, so that in this derivation we can reverse the roles and x and y. In other words, in the same manner we prove that $\partial^2 z/\partial x\,\partial y$ is given by the same limit, and so we obtain the desired result: the mixed partials are equal. ■

Supplement to Section 15.1: Partial Derivatives and Wave Motion

Two of the most important problems in the historical development of partial differentiation concerned *wave motion* and *heat conduction*. Here we concentrate on the first of these problems. (See also Exercises 71 and 72.)

Consider water in motion in a narrow tank, as illustrated in Fig. 15.1.3. We will assume that the motion of the water is gentle enough so that, at any instant of time, the height z of the water above the bottom of the tank is a function of the position y measured along the long direction of the tank; this means that there are no "breaking waves" and that the height of water is constant along the short direction of the tank. Since the water is in motion, the height z depends on the time as well as on y, so we may write $z = f(t, y)$; the domain of the function f consists of all pairs (t, y) such that t lies in the interval of time relevant for the experiment, and $a \leqslant y \leqslant b$, where a and b mark the ends of the tank.

We can graph the entire function f as a surface in (t, y, z) space lying over the strip $a \leqslant y \leqslant b$ (see Fig. 15.1.4); the section of this surface by a plane of the form $t = t_0$ is a curve which shows the configuration of the water at the moment t_0 (such as each of the "snapshots" in Fig. 15.1.3). This curve is the graph of a function of *one* variable, $z = g(y)$, where g is defined by $g(y) = f(t_0, y)$. If we take the derivative of the function g at a point y_0 in (a, b), we get a number $g'(y_0)$ which represents the slope of the water's surface at the time t_0 and at the location y_0. (See Fig. 15.1.5.) It could be observed as the slope of a small stick parallel to the sides of the tank floating on the water at that time and position.

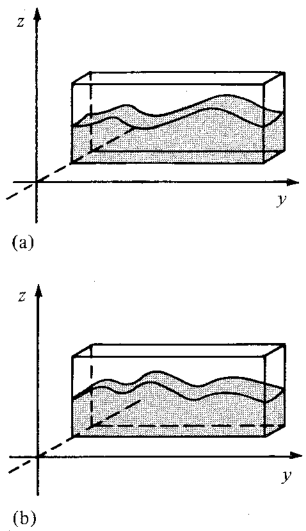

Figure 15.1.3. Moving water in a narrow tank shown at two different instants of time.

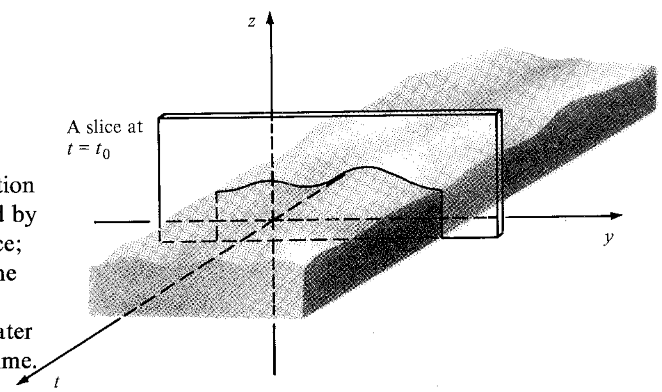

Figure 15.1.4. The motion of the water is depicted by a graph in (t, y, z) space; sections by planes of the form $t = t_0$ show the configuration of the water at various instants of time.

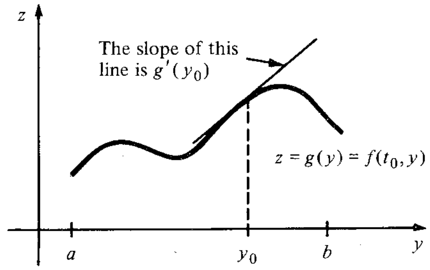

Figure 15.1.5. The derivative $g'(y_0)$ of the function $g(y) = f(t_0, y)$ represents the slope of the water's surface at time t_0 and position y_0.

This number, $g'(y_0)$, is obtained from the function f by:
1. Fixing t at the value t_0.
2. Differentiating the resulting function of y.
3. Setting y equal to y_0.

The number $g'(y_0)$ is just $f_y(t_0, y_0)$.

We can also define the partial derivative of f with respect to t at (t_0, y_0); it is obtained by:
1. Fixing y at the value y_0.
2. Differentiating the resulting function of t.
3. Setting t equal to t_0.

The result is $f_t(t_0, y_0)$. In the first step, we obtain the function $h(t) = f(t, y_0)$, which represents the vertical motion of the water's surface observed at the fixed position y_0. The derivative with respect to t is, therefore, the *vertical velocity* of the surface at the position y_0. It could be observed as the vertical velocity of a cork floating on the water at that position. Finally, setting t equal to t_0 merely involves observing the velocity at the specific time t_0.

Exercises for Section 15.1

Compute f_x and f_y for the functions in Exercises 1–8 and evaluate them at the indicated points.

1. $f(x, y) = xy$; $(1, 1)$
2. $f(x, y) = x/y$; $(1, 1)$
3. $f(x, y) = \tan^{-1}(x - 3y^2)$; $(1, 0)$
4. $f(x, y) = \sqrt{x^2 + y^2}$; $(1, -1)$
5. $f(x, y) = e^{xy}\sin(x + y)$; $(0, 0)$
6. $f(x, y) = \ln(x^2 + y^2 + 1)$; $(0, 0)$
7. $f(x, y) = 1/(x^3 + y^3)$; $(-1, 2)$
8. $f(x, y) = e^{-x^2 - y^2}$; $(1, -1)$

Compute f_x, f_y, and f_z for the functions in Exercises 9–12, and evaluate them at the indicated points.

9. $f(x, y, z) = xyz$; $(1, 1, 1)$
10. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; $(3, 0, 4)$
11. $f(x, y, z) = \cos(xy^2) + e^{3xyz}$; $(\pi, 1, 1)$
12. $f(x, y, z) = x^{yz}$; $(1, 1, 0)$

Find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ for the functions in Exercises 13–16.

13. $z = 3x^2 + 2y^2$
14. $z = \sin(x^2 - 3xy)$
15. $z = (2x^2 + 7x^2y)/3xy$
16. $z = x^2y^2e^{2xy}$

Find the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ in Exercises 17–20.

17. $u = e^{-xyz}(xy + xz + yz)$
18. $u = \sin(xy^2z^3)$
19. $u = e^x\cos(yz^2)$
20. $u = (xy^3 + e^z)/(x^3y - e^z)$

Compute the indicated partial derivatives in Exercises 21–24.

21. $\dfrac{\partial}{\partial y}\left(\dfrac{xe^y - 1}{ye^x + 1}\right)$
22. $\dfrac{\partial}{\partial u}(uvw - \sin(uvw))$
23. $\dfrac{\partial}{\partial b}(mx + b^2)^8$
24. $\dfrac{\partial}{\partial m}(mx + b^2)^8$

In Exercises 25–28, let

$$f(x, y) = 3x^2 + 2\sin(x/y^2) + y^3(1 - e^x)$$

and find the indicated quantities.

25. $f_x(2, 3)$
26. $f_x(0, 1)$
27. $f_y(1, 1)$
28. $f_y(-1, -1)$
29. Let $z = (\sin x)e^{-xy}$.
    (a) Find $\partial z/\partial y$.
    (b) Evaluate $\partial z/\partial y$ at the following four points: $(0, 0)$, $(0, \pi/2)$, $(\pi/2, 0)$, and $(\pi/2, \pi/2)$.
30. Let $u = (xy/z)\cos(yz)$.
    (a) Find $\partial u/\partial z$.
    (b) Evaluate $\partial u/\partial z$ at $(1, \pi, 1)$, $(0, \pi/2, 1)$, and $(1, \pi, 1/2)$.

Let $g(t, u, v) = \ln(t + u + v) - \tan(tuv)$ and find the indicated quantities in Exercises 31–36.

31. $g_t(0, 0, 1)$
32. $g_t(1, 0, 0)$
33. $g_u(1, 2, 3)$
34. $g_u(2, 3, 1)$
35. $g_v(-1, 3, 5)$
36. $g_v(-1, 5, 3)$

In Exercises 37–40, compute the indicated partial derivatives

37. $\dfrac{\partial}{\partial s} e^{stu^2}$
38. $\dfrac{\partial}{\partial r}\left(\dfrac{1}{3}\pi r^2h\right)$
39. $\dfrac{\partial}{\partial \lambda}\left(\dfrac{\cos \lambda\mu}{1 + \lambda^2 + \mu^2}\right)$
40. $\dfrac{\partial}{\partial a}(bcd)$
41. If $f(x, y, z)$ is a function of three variables, express f_z as a limit.
42. Find
$$\lim_{\Delta y \to 0} \frac{3 + (x + y + \Delta y)^2z - (3 + (x + y)^2z)}{\Delta y}$$
43. In the situation of Example 4, how fast is the temperature changing if we proceed directly west? (The longitude y is increasing as we go west.)

44. Chicago Skate Company produces three kinds of roller skates. The cost in dollars for producing x, y, and z units of each, respectively, is $c(x, y, z) = 3000 + 27x + 36y + 47z$.
    (a) The value of $\partial c/\partial x$ is the change in cost due to a one unit increase in production of the least expensive skate, the levels of production of the higher-priced units being held fixed. Find it.
    (b) Find $\partial c/\partial z$, and interpret.
45. If three resistors R_1, R_2, and R_3 are connected in parallel, the total electrical resistance is determined by the equation
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$
    (a) What is $\partial R/\partial R_1$?
    (b) Suppose that R_1, R_2, and R_3 are variable resistors set at 100, 200, and 300 ohms, respectively. How fast is R changing with respect to R_1?
46. Consider the topographical map of Yosemite Valley in Fig. 14.R.3. Let r represent the east–west coordinate on the map, increasing from west to east. Let s be the north–south coordinate, increasing as you go north. (East is the positive r direction, north the positive s direction.) Let h be the elevation above sea level.
    (a) Explain how $\partial h/\partial r$ and $\partial h/\partial s$ are related to the distances between contour lines and their directions.
    (b) At the center of the letter o in Half Dome, what is the sign of $\partial h/\partial r$? of $\partial h/\partial s$?

In Exercises 47–50, find the partial derivatives $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\,\partial y$, $\partial^2 z/\partial y\,\partial x$ and $\partial^2 z/\partial y^2$ for each of the functions in the indicated exercise.

47. Exercise 13
48. Exercise 14
49. Exercise 15
50. Exercise 16
51. Let $f(x, y, z) = x^2y + xy^2 + yz^2$. Find f_{xy}, f_{yz}, f_{zx}, and f_{xyz}.
52. Let $z = x^4y^3 - x^8 + y^4$.
    (a) Compute $\partial^3 z/\partial y\,\partial x\,\partial x$, $\partial^3 z/\partial x\,\partial y\,\partial x$, and $\partial^3 z/\partial x\,\partial x\,\partial y$.
    (b) Compute $\partial^3 z/\partial x\,\partial y\,\partial y$, $\partial^3 z/\partial y\,\partial x\,\partial y$, and $\partial^3 z/\partial y\,\partial y\,\partial x$.

Compute $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y\,\partial x$, $\partial^2 u/\partial y^2$, and $\partial^2 u/\partial x\,\partial y$ for each of the functions in Exercises 53–56. Check directly the equality of mixed partials.

53. $u = 2xy/(x^2 + y^2)^2$
54. $u = \cos(xy^2)$
55. $u = e^{-xy^2} + y^3x^4$
56. $u = 1/(\cos^2 x + e^{-y})$
57. Prove, using ε's and δ's, that $\lim_{(x,y)\to(x_0,y_0)} y = y_0$.
58. Prove, using ε's and δ's, that $\lim_{(x,y)\to(x_0,y_0)}(x + y) = x_0 + y_0$.

In Exercises 59–66, evaluate the given limits if they exist (do not attempt a precise justification).

59. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y + y^3}{x^2 + y^2}$
60. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x + y}{(x-1)^2 + 1}$
61. $\displaystyle\lim_{(x,y)\to(2,3)} \frac{x^2y + y^3 + x^2 + y^2}{\sqrt{x^2 + y^2}}$
62. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2 + 3y^2 + x^3y^3}{x^2 + y^2 + x^4y^4}$
63. $\displaystyle\lim_{(x,y)\to(1,1)} e^x\cos(\pi y)$
64. $\displaystyle\lim_{(x,y)\to(0,1)} e^{xy}\cos(\pi xy)$
65. $\displaystyle\lim_{(x,y)\to(0,0)} \sin(xy)$
66. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{1 + \ln(1 + 1/(x^2 + y^2))}$
67. Let $f(x, y) = x^2 + y^2$ and suppose that (x, y) moves along the curve $(x(t), y(t)) = (\cos t, e^t)$.
    (a) Find $g(t) = f(x(t), y(t))$ and use your formula to compute $g'(t_0)$.
    (b) Show that this is the same as
$$f_x(x(t_0), y(t_0)) \cdot x'(t_0) + f_y(x(t_0), y(t_0)) \cdot y'(t_0).$$
68. Let $f(x, y, z) = x^2 + 2y - z$ and suppose the point (x, y, z) moves along the parametric curve $(1, t, t^2)$.
    (a) Let $g(t) = f(1, t, t^2)$ and compute $g'(t)$.
    (b) Show that your answer in (a) is equal to
$$f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt}.$$
69. A function $u = f(x, y)$ with continuous second partial derivatives satisfying *Laplace's equation*
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
is called a *harmonic function*. Show that the function $u(x, y) = x^3 - 3xy^2$ is harmonic.
70. Which of the following functions satisfy Laplace's equation? (See Exercise 69).
    (a) $f(x, y) = x^2 - y^2$;
    (b) $f(x, y) = x^2 + y^2$;
    (c) $f(x, y) = xy$;
    (d) $f(x, y) = y^3 + 3x^2y$;
    (e) $f(x, y) = \sin x \cosh y$;
    (f) $f(x, y) = e^x \sin y$.
71. Let f and g be differentiable functions of one variable. Set $\varphi = f(x - t) + g(x + t)$.
    (a) Prove that φ satisfies the *wave equation*: $\partial^2\varphi/\partial t^2 = \partial^2\varphi/\partial x^2$.
    (b) Sketch the graph of φ against t and x if $f(x) = x^2$ and $g(x) = 0$.

72. (a) Show that function $g(x,t) = 2 + e^{-t}\sin x$ satisfies the *heat equation*: $g_t = g_{xx}$. (Here $g(x,t)$ represents the temperature in a rod at position x and time t.)
(b) Sketch the graph of g for $t \geqslant 0$. [*Hint*: Look at sections by the planes $t = 0$, $t = 1$, and $t = 2$.]
(c) What happens to $g(x,t)$ as $t \to \infty$? Interpret this limit in terms of the behavior of heat in a rod.

73. The productivity z per employee per week of a company depends on the size x of the labor force and the amount y of investment capital in millions of dollars. A typical formula is $z(x, y) = 60xy - x^2 - 4y^2$.
(a) The value of $\partial z/\partial x$ at $x = 5$, $y = 3$ is the marginal productivity of labor per worker at a labor force of 5 people and investment level of 3 million dollars. Find it.
(b) Find $\partial z/\partial y$ at $x = 5$, $y = 3$, and interpret.

74. The productivity z of a company is given by $z(x, y) = 100xy - 2x^2 - 6y^2$ where $x \times 10^3$ people work for the company, and the capital investment of the company is y million dollars.
(a) Find the marginal productivity of labor $\partial z/\partial x$. This number is the expected change in production for an increase of 1000 staff with fixed capital investment.
(b) Find $\partial z/\partial y$ when $x = 5$ and $y = 3$. Interpret.

★75. Show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{1/3}$$

does not exist.

★76. Let

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0,0), \\ 0, & (x, y) = (0,0). \end{cases}$$

(a) If $(x, y) \neq (0,0)$, compute f_x and f_y.
(b) What is the value of $f(x,0)$ and $f(0, y)$?
(c) Show that $f_x(0,0) = 0 = f_y(0,0)$.

★77 Consider the function f in Exercise 76.
(a) Show that $f_x(0, y) = -y$ when $y \neq 0$.
(b) What is $f_y(x,0)$ when $x \neq 0$?
(c) Show that $f_{yx}(0,0) = 1$ and $f_{xy}(0,0) = -1$. [*Hint*: Express them as limits.]
(d) What went wrong? Why are the mixed partials not equal?

★78. Suppose that f is continuous at (x_0, y_0) and $f(x_0, y_0) > 0$. Show that there is a disk about (x_0, y_0) on which $f(x, y) > 0$.

15.2 Linear Approximations and Tangent Planes

The plane tangent to the graph of a function of two variables has two slopes.

In the calculus of functions of one variable, the simplest functions are the linear functions $l(x) = mx + b$. The derivative of such a function is the constant m, which is the slope of the graph or the rate of change of y with respect to x. If $f(x)$ is any differentiable function, its tangent line at x_0 is the graph of the *linear approximation* $y = f(x_0) + f'(x_0)(x - x_0)$.

To extend these ideas to functions of two variables, we begin by looking at linear functions of the form $z = l(x, y) = ax + by + c$, whose graphs are planes. Such a plane has two "slopes," the numbers a and b, which determine the direction of its normal vector $-a\mathbf{i} - b\mathbf{j} + \mathbf{k}$ (see Section 13.4). These slopes can be recovered from the function l as the partial derivatives $l_x = a$ and $l_y = b$. By analogy with the situation for one variable, we define the *linear approximation* at (x_0, y_0) for a general function f of two variables to be the linear function

$$l(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which is of the form $ax + by + c$ with

$$a = f_x(x_0, y_0), \quad b = f_y(x_0, y_0), \quad \text{and}$$

$$c = f(x_0, y_0) - x_0 f_x(x_0, y_0) - y_0 f_y(x_0, y_0).$$

The function l is the unique linear function which has at (x_0, y_0) the same value *and* the same partial derivatives as f. The graph

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{1}$$

of the linear approximation is a plane through $(x_0, y_0, f(x_0, y_0))$, with normal vector $-f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}$; it is called the *tangent plane* at (x_0, y_0) to the graph of f (see Fig. 15.2.1).

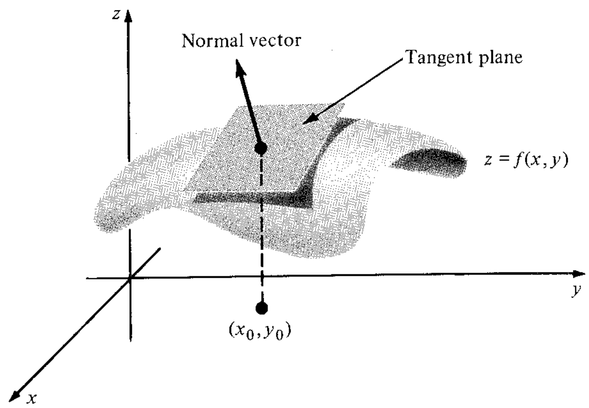

Figure 15.2.1. The tangent plane at (x_0, y_0) to the graph $z = f(x, y)$ has the equation $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. A vector normal to the plane is

$$\mathbf{n} = -f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}.$$

Example 1 Find the equation of the plane tangent to the hemisphere $z = \sqrt{1 - x^2 - y^2}$ at a point (x_0, y_0). Interpret your result geometrically.

Solution Letting $f(x, y) = \sqrt{1 - x^2 - y^2}$, we have $f_x(x, y) = -x/\sqrt{1 - x^2 - y^2}$ and $f_y(x, y) = -y/\sqrt{1 - x^2 - y^2}$. The equation of the tangent plane at (x_0, y_0, z_0) is obtained from (1) to be

$$z = \sqrt{1 - x_0^2 - y_0^2} - \frac{x_0}{\sqrt{1 - x_0^2 - y_0^2}}(x - x_0) - \frac{y_0}{\sqrt{1 - x_0^2 - y_0^2}}(y - y_0),$$

$$\text{or} \quad z = z_0 - \frac{x_0}{z_0}(x - x_0) - \frac{y_0}{z_0}(y - y_0).$$

A normal vector is thus $\frac{x_0}{z_0}\mathbf{i} + \frac{y_0}{z_0}\mathbf{j} + \mathbf{k}$. Multiplying by z_0, we find that another normal vector is $x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. Thus we have recovered the geometric result that the tangent plane at a point P of a sphere is perpendicular to the vector from the center of the sphere to P. ▲

The linear approximation may be defined as well for a function of three variables. We include its definition in the following box.

Linear Approximation and Tangent Plane

The linear approximation at (x_0, y_0) of $f(x, y)$ is the linear function:

$$l(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{2}$$

The graph $z = l(x, y)$ is called the *tangent plane* to the graph of f at (x_0, y_0). It has normal vector $-f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}$.

The linear approximation at (x_0, y_0, z_0) to $f(x, y, z)$ is the linear function:

$$\begin{aligned} l(x, y, z) = f(x_0, y_0, z_0) &+ f_x(x_0, y_0, z_0)(x - x_0) \\ &+ f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0). \end{aligned} \tag{3}$$

Example 2 Find the equation of the plane tangent to the graph of

$$f(x, y) = (x^2 + y^2)/xy$$

at $(x_0, y_0) = (1, 2)$.

Solution Here $x_0 = 1$, $y_0 = 2$, and $f(1,2) = \frac{5}{2}$. The partial derivative with respect to x is

$$f_x(x, y) = \frac{2x \cdot xy - (x^2 + y^2)y}{(xy)^2} = \frac{x^2y - y^3}{(xy)^2} = \frac{x^2 - y^2}{x^2y},$$

which is $-\frac{3}{2}$ at $(1,2)$. Similarly,

$$f_y(x, y) = \frac{y^2x - x^3}{(xy)^2} = \frac{y^2 - x^2}{xy^2},$$

which is $\frac{3}{4}$ at $(1,2)$. Thus the tangent plane is given by the equation (1)

$$z = -\tfrac{3}{2}(x - 1) + \tfrac{3}{4}(y - 2) + \tfrac{5}{2},$$

i.e., $4z = -6x + 3y + 10$. ▲

Example 3 Find a formula for a unit normal vector to the graph of the function $f(x, y) = e^xy$ at the point $(-1, 1)$.

Solution Since a normal vector is $-f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}$, a unit normal is obtained by normalizing:

$$\mathbf{n} = \frac{-f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}.$$

In this case, $f_x(x, y) = e^xy$ and $f_y(x, y) = e^x$. Evaluating the partial derivatives at $(-1, 1)$, we find a normal to be $-e^{-1}\mathbf{i} - e^{-1}\mathbf{j} + \mathbf{k}$, and so a unit normal is

$$\frac{-e^{-1}\mathbf{i} - e^{-1}\mathbf{j} + \mathbf{k}}{\sqrt{e^{-2} + e^{-2} + 1}} = \frac{-e^{-1}}{\sqrt{2e^{-2} + 1}}\mathbf{i} - \frac{e^{-1}}{\sqrt{2e^{-2} + 1}}\mathbf{j} + \frac{1}{\sqrt{2e^{-2} + 1}}\mathbf{k}. \ ▲$$

Just as in one-variable calculus, we can use the linear approximation for approximate numerical computations. Suppose that the number $z = f(x, y)$ depends on both x and y and we want to know how much z changes as x and y are changed a little. The partial derivative $f_x(x_0, y_0)$ gives the rate of change of z with respect to x at (x_0, y_0). Thus the change in z which results from a change Δx in x should be about

$$f_x(x_0, y_0)\,\Delta x.$$

Similarly, the change in z caused by a shift in y by Δy should be about

$$f_y(x_0, y_0)\,\Delta y.$$

Thus the total change in z should be approximately

$$\Delta z \approx f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y. \tag{4}$$

Notice that the change in z is obtained by simply *adding* the changes due to Δx and Δy. If we write $\Delta x = x - x_0$ and $\Delta y = y - y_0$, then the expression for Δz is the linear approximation to $f(x, y) - f(x_0, y_0)$ at (x_0, y_0).

Example 4 Calculate an approximate value for $(0.99e^{0.02})^8$. Compare with the value from a calculator.

Solution Let $z = f(x, y) = (xe^y)^8$ and let $x_0 = 1$ and $y_0 = 0$, so $f(1,0) = 1$. We get

$$\frac{\partial z}{\partial x} = 8x^7e^{8y}, \qquad \text{which is 8 at } (1,0),$$

and

$$\frac{\partial z}{\partial y} = 8x^8e^{8y}, \qquad \text{which is 8 at } (1,0).$$

Thus if we let $x = 0.99$ and $y = 0.02$ so that $x - x_0 = -0.01$ and $y - y_0 = 0.02$, the linear approximation is (by (2) or (4))

$$1 + 8(-0.01) + 8(0.02) = 1.08.$$

The value for $(0.99e^{0.02})^8$ obtained on our calculator is 1.082850933. ▲

Example 5 Find an approximate value for $\sin(0.01) \cdot \cos(0.99\pi)$.

Solution Let $f(x, y) = \sin x \cos y$ and $x_0 = 0$, $y_0 = \pi$. Then if $x = 0.01$ and $y = 0.99\pi$,

$$\begin{aligned} f(x, y) &\approx f_x(0,\pi)(x - 0) + f_y(0,\pi)(y - \pi) + f(0,\pi) \\ &= -1(0.01) + 0 + 0 \\ &= -0.01. \end{aligned}$$

(The value of $f(x, y)$ computed on our calculator is -0.009994899.) ▲

Example 6 A multiplication problem is altered by taking a small amount from one factor and adding it to the other. How can you tell whether the product increases or decreases?

Solution Let $f(x, y) = xy$. If the amount moved from y to x is h, we must look at

$$f(x + h, y - h) - f(x, y),$$

which may be approximated by

$$hf_x(x, y) + (-h)f_y(x, y).$$

The partial derivatives are $f_x(x, y) = y$ and $f_y(x, y) = x$, so the linear approximation to the change in the product is $h(y - x)$. Thus, the product increases when the increment h is taken from the larger factor. ▲

Exercises for Section 15.2

Find equations for the planes tangent to the surfaces in Exercises 1–4 at the indicated points.

1. $z = x^3 + y^3 - 6xy$; $(1, 2, -3)$
2. $z = (\cos x)(\cos y)$; $(0, \pi/2, 0)$
3. $z = (\cos x)(\sin y)$; $(0, \pi/2, 1)$
4. $z = 1/xy$; $(1, 1, 1)$

Find the equation of the plane tangent to the graph of $z = f(x, y) = x^2 + 2y^3 + 1$ at the points in Exercises 5–8.

5. $(1, 1, 4)$
6. $(-1, -1, 0)$
7. $(0, 0, 1)$
8. $(1, -1, 0)$

Find the equation of the tangent plane of the graph of f at the point $(x_0, y_0, f(x_0, y_0))$ for the functions and points in Exercises 9–12.

9. $f(x, y) = x - y + 2$; $(x_0, y_0) = (1, 1)$.
10. $f(x, y) = x^2 + 4y^2$; $(x_0, y_0) = (2, -1)$.
11. $f(x, y) = xy$; $(x_0, y_0) = (1, 1)$.
12. $f(x, y) = x/(x + y)$; $(x_0, y_0) = (1, 0)$.

For each of the indicated functions and points in Exercises 13–16, find a unit normal vector to the graph at $(x_0, y_0, f(x_0, y_0))$.

13. f and (x_0, y_0) as in Exercise 9.
14. f and (x_0, y_0) as in Exercise 10.
15. f and (x_0, y_0) as in Exercise 11.
16. f and (x_0, y_0) as in Exercise 12.

Find an appropriate value for each of the quantities in Exercises 17–22 using the linear approximation.

17. $(1.01)^2[1-\sqrt{1.98}]$ [*Hint*: $1.96=(1.4)^2$.]
18. $\tan\left(\dfrac{\pi+0.01}{3.97}\right)$
19. $(0.99)^3+(2.01)^3-6(0.99)(2.01)$
20. $(0.98)\sin\left(\dfrac{0.99}{1.03}\right)$
21. $(0.98)(0.99)(1.03)$
22. $\sqrt{(4.01)^2+(3.98)^2+(2.02)^2}$
23. In the setup of Example 4, Section 15.1, at Dawson Creek, is the temperature increasing or decreasing as you proceed south? As you proceed east? Southeast?
24. Refer to Exercise 45, Section 15.1. If, in part (b), R_1 is increased by 1 ohm, R_2 is decreased by 2 ohms, and R_3 is increased by 4 ohms, use the linear approximation to calculate the change in R. Compare with a direct calculation on a calculator.
25. Let $f(a,v)$ be the length of a side of a cube whose surface area is a and whose volume is v. Find the linear approximation to $f(6+\Delta a, 1+\Delta v)$.
26. Let $g(u,v)$ be the gas mileage if you drive u miles and use v gallons of gasoline. How does $g(u,v)$ change if you go Δu extra miles on Δv extra gallons? (Use the linear approximation.)
27. Suppose that $z=f(x,y)=x^2+y^2$.
    (a) Find $\partial z/\partial y|_{(1,1)}$
    (b) Describe the curve obtained by intersecting the graph of f with the plane $x=1$.
    (c) Find a tangent vector to this curve at the point $(1,1,f(1,1))$.
28. Repeat Exercise 27 for $z=f(x,y)=e^{xy}$.
29. Let $f(x,y)=-(1-x^2-y^2)^{1/2}$ for (x,y) such that $x^2+y^2<1$. Show that the plane tangent to the graph of f at $(x_0,y_0,f(x_0,y_0))$ is orthogonal to the vector with components $(x_0,y_0,f(x_0,y_0))$. Interpret this geometrically.

★30. (a) Let k be a differentiable function of one variable, and let $f(x,y)=k(xy)$. Suppose that x and y are functions of t: $x=g(t)$, $y=h(t)$, and set $F(t)=f(g(t),h(t))$. Prove that

$$F'(t)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}.$$

(b) If $f(x,y)=k(x)l(y)$, show that the formula in (a) is still valid. (These are special cases of the chain rule, proved in the next section.)

15.3 The Chain Rule

The derivative of a composite function with several intermediate variables is a sum of products.

In Chapter 2 we developed the chain rule for functions of one variable: If y is a function of x and z is a function of y, then z also may be regarded as a function of x, and

$$\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}.$$

For functions of several variables, the chain rule is more complicated. First we consider the case where z is a function of x and y, and x and y are functions of t; we can then regard z as a function of t. In this case the chain rule states that

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}.$$

The chain rule applies when quantities in which we are interested depend in a known way upon other quantities which in turn depend upon a third set of quantities. Suppose, for example, that the temperature T on the surface of a pond is a function $f(x,y)$ of the position coordinates (x,y). If a duck swims on the pond according to the parametric equations $x=g(t)$, $y=h(t)$, it will feel the water temperature varying with time according to the function $T=F(t)=f(g(t),h(t))$. The rate at which this temperature changes with respect to time is the derivative dT/dt. By analogy with the chain rule in one variable, we may expect this derivative to depend upon the direction and magnitude of the duck's velocity, as given by the derivatives dx/dt and dy/dt,

as well as upon the partial derivatives $\partial T/\partial x$ and $\partial T/\partial y$ of temperature with respect to position. The correct formula relating all these derivatives is given in the following box. The formula will be proved after we see how it works in an example.

The Chain Rule

To find dz/dt, when $z = f(x, y)$ has continuous partial derivatives and $x = g(t)$ and $y = h(t)$ are differentiable, multiplying the partial derivatives of z with respect to each of the intermediate variables x and y by the derivative of that intermediate variable with respect to t, and add the products: If $F(t) = f(g(t), h(t))$, then

$$F'(t) = f_x(g(t), h(t))\,g'(t) + f_y(g(t), h(t))h'(t).$$

In Leibniz notation,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\,.$$

For three intermediate variables, if u depends on x, y, and z, and x, y, and z depend on t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}\,.$$

Example 1 Suppose that a duck is swimming in a circle, $x = \cos t$, $y = \sin t$, while the water temperature is given by the formula $T = x^2e^y - xy^3$. Find dT/dt: (a) by the chain rule; (b) by expressing T in terms of t and differentiating.

Solution (a) $\partial T/\partial x = 2xe^y - y^3$; $\partial T/\partial y = x^2e^y - 3xy^2$; $dx/dt = -\sin t$; $dy/dt = \cos t$. By the chain rule, $dT/dt = (\partial T/\partial x)(dx/dt) + (\partial T/\partial y)(dy/dt)$, so

$$\frac{dT}{dt} = (2xe^y - y^3)(-\sin t) + (x^2e^y - 3xy^2)\cos t$$

$$= (2\cos t\, e^{\sin t} - \sin^3 t)(-\sin t) + (\cos^2 t\, e^{\sin t} - 3\cos t\sin^2 t)\cos t$$

$$= -2\cos t\sin t\, e^{\sin t} + \sin^4 t + \cos^3 t\, e^{\sin t} - 3\cos^2 t\sin^2 t.$$

(b) Substituting for x and y in the formula for T gives

$$T = \cos^2 t e^{\sin t} - \cos t\sin^3 t,$$

and differentiating this gives

$$\frac{dT}{dt} = 2\cos t(-\sin t)e^{\sin t} + \cos^2 t\, e^{\sin t}\cos t + \sin t\sin^3 t - (\cos t)3\sin^2 t\cos t$$

$$= -2\cos t\sin t\, e^{\sin t} + \cos^3 t\, e^{\sin t} + \sin^4 t - 3\cos^2 t\sin^2 t,$$

which is the same as the answer in part (a). ▲

An intuitive argument for the chain rule is based on the linear approximation of Section 15.2. If the position of the duck changes from the point (x, y) to the point $(x + \Delta x, y + \Delta y)$, the temperature change ΔT is given approximately by $(\partial T/\partial x)\Delta x + (\partial T/\partial y)\Delta y$. On the other hand, the linear approximation for functions of one variable gives $\Delta x \approx (dx/dt)\Delta t$ and $\Delta y \approx (dy/dt)\Delta t$. Putting these two approximations together gives

$$\Delta T \approx \frac{\partial T}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial T}{\partial y}\frac{dy}{dt}\Delta t.$$

Hence

$$\frac{\Delta T}{\Delta t} \approx \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}. \tag{1}$$

As $\Delta t \to 0$, the approximations become more and more accurate and the ratio $\Delta T/\Delta t$ approaches dT/dt, so the approximation formula (1) becomes the chain rule. The argument for three variables is similar.

In trying to make a proof out of this intuitive argument, one discovers that more than differentiability is required; the partial derivatives of T should be continuous. The technical details are outlined in Exercise 20.

Example 2 Verify the chain rule for $u = xe^{yz}$ and $(x, y, z) = (e^t, t, \sin t)$.

Solution Substituting the formulas for x, y, and z in the formula for u gives

$$u = e^t \cdot e^{t\sin t} = e^{t(1+\sin t)},$$

$$\text{so} \quad \frac{du}{dt} = \left[t\cos t + (1+\sin t)\right]e^{t(1+\sin t)}.$$

The chain rule says that this should equal

$$\begin{aligned}\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} &= e^{yz}e^t + xze^{yz}\cdot 1 + xye^{yz}\cos t \\ &= e^{t\sin t}e^t + e^t\sin t\, e^{t\sin t} + e^t\cdot t\cdot e^{t\sin t}\cos t \\ &= e^{t(1+\sin t)}(1+\sin t + t\cos t),\end{aligned}$$

which it does. ▲

Example 3 Suppose that $v = r\cos(st) - e^s\sin(rt)$ and that r, s, and t are functions of x. Find an expression for dv/dx.

Solution We use the chain rule with a change of notation. If $v = f(r,s,t)$ and r,s,t are functions of x, then

$$\frac{dv}{dx} = \frac{\partial v}{\partial r}\frac{dr}{dx} + \frac{\partial v}{\partial s}\frac{ds}{dx} + \frac{\partial v}{\partial t}\frac{dt}{dx}.$$

In this case we get

$$\begin{aligned}\frac{dv}{dx} &= \left[\cos(st) - te^s\cos(rt)\right]\frac{dr}{dx} - \left[tr\sin(st) + e^s\sin(rt)\right]\frac{ds}{dx} \\ &\quad - \left[rs\sin(st) + re^s\cos(rt)\right]\frac{dt}{dx}.\end{aligned}$$ ▲

Example 4 What do you get if you apply the chain rule to the case $z = xy$, where x and y are arbitrary functions of t?

Solution If $z = xy$, then $\partial z/\partial x = y$ and $\partial z/\partial y = x$, so the chain rule gives $dz/dt = y(dx/dt) + x(dy/dt)$, which is precisely the product rule for functions of one variable. ▲

The chain rule for the case of two intermediate variables has a nice geometric interpretation involving the tangent plane. Recall from Section 15.2 that the tangent plane to the graph $z = f(x, y)$ at the point (x_0, y_0) is given by the linear equation $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. For this formula to be consistent with the definition of the tangent line to a curve, we would like the following statement to be true.

Tangents to Curves in Graphs

If $(x, y, z) = (g(t), h(t), k(t))$ is any curve on the surface $z = f(x, y)$ with $(g(t_0), h(t_0)) = (x_0, y_0)$, then the tangent line to the curve at t_0 lies in the tangent plane to the surface at (x_0, y_0). (In this statement, all derivatives are assumed to be continuous.)

To verify the above statement, we start with the fact that $(g(t), h(t), k(t))$ lies on the surface $z = f(x, y)$, i.e.,

$$k(t) = f(g(t), h(t)).$$

Differentiate both sides using the chain rule and then set $t = t_0$:

$$k'(t_0) = f_x(x_0, y_0)g'(t_0) + f_y(x_0, y_0)h'(t_0);$$

but this shows that the tangent line

$$z - z_0 = tk'(t_0), \qquad x - x_0 = tg'(t_0), \qquad y - y_0 = th'(t_0)$$

satisfies $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$; that is, the tangent line lies in the tangent plane.

You may think of the preceding box as the "geometric statement" of the chain rule. It is illustrated in Fig. 15.3.1.

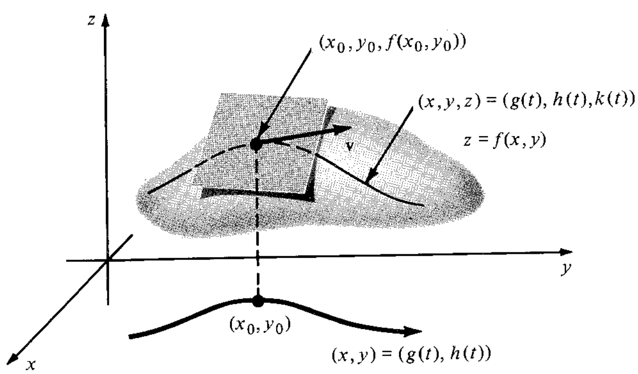

Figure 15.3.1. If a curve lies on the surface $z = f(x, y)$, then the tangent line (with direction vector **v**) to the curve lies in the tangent plane of the surface.

Example 5 Show that for any curve $\boldsymbol{\sigma}(t)$ in the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, the velocity vector $\boldsymbol{\sigma}'(t)$ is perpendicular to $\boldsymbol{\sigma}(t)$.

Solution Let $(x, y, z) = \boldsymbol{\sigma}(t)$. By the preceding box, $\boldsymbol{\sigma}'(t)$ is perpendicular to the normal vector to the hemisphere at (x, y, z). From Example 1 of the previous section, this normal vector is just $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \boldsymbol{\sigma}(t)$. Thus, $\boldsymbol{\sigma}'(t)$ is perpendicular to $\boldsymbol{\sigma}(t)$. ▲

Example 6 Show that the tangent plane at each point (x_0, y_0, z_0) of the cone $z = \sqrt{x^2 + y^2}$ $((x, y) \neq (0,0))$ contains the line passing through (x_0, y_0, z_0) and the origin.

Solution The line l through (x_0, y_0, z_0) and the origin has parametrization $(x, y, z) = (x_0t, y_0t, z_0t)$. Since this line lies in the cone for all $t > 0$, $(z^2 = z_0^2t^2 = (x_0^2 + y_0^2)t^2 = x^2 + y^2)$, the geometric interpretation of the chain rule implies that the tangent plane to the cone contains the tangent line to l; but the tangent line to l is l itself, so l is contained in the tangent plane. ▲

Exercises for Section 15.3

1. Suppose that a duck is swimming in a straight line $x = 3 + 8t$, $y = 3 - 2t$, while the water temperature is given by the formula $T = x^2\cos y - y^2\sin x$. Find dT/dt in two ways: (a) by the chain rule and (b) by expressing T in terms of t and differentiating.
2. Suppose that a duck is swimming along the curve $x = (3 + t)^2$, $y = 2 - t^2$, while the water temperature is given by the formula $T = e^x(y^2 + x^2)$. Find dT/dt in two ways: (a) by the chain rule and (b) by expressing T in terms of t and differentiating.

Verify the chain rule for the functions and curves in Exercises 3–6.

3. $f(x, y) = (x^2 + y^2)\ln(\sqrt{x^2 + y^2})$; $\boldsymbol{\sigma}(t) = (e^t, e^{-t})$.
4. $f(x, y) = xe^{x^2+y^2}$; $\boldsymbol{\sigma}(t) = (t, -t)$.
5. $f(x, y, z) = x + y^2 + z^3$; $\boldsymbol{\sigma}(t) = (\cos t, \sin t, t)$.
6. $f(x, y, z) = e^{x-z}(y^2 - x^2)$; $\boldsymbol{\sigma}(t) = (t, e^t, t^2)$.
7. Verify the chain rule for $u = x/y + y/z + z/x$, $x = e^t$, $y = e^{t^2}$, $z = e^{t^3}$.
8. Verify the chain rule for $u = \sin(xy)$, $x = t^2 + t$, $y = t^3$.
9. Show that applying the chain rule to $z = x/y$ (where x and y are arbitrary functions of t) gives the quotient rule for functions of one variable.
10. (a) Apply the chain rule to $u = xyz$, where x, y, and z are functions of t, to get a rule for differentiating a product of three functions of one variable.
    (b) Derive the rule in (a) by using one-variable calculus.
11. Let $z = \sqrt{x^2 + y^2} + 2xy^2$, where x and y are functions of u. Find an expression for dz/du.
12. If $u = \sin(a + \cos b)$, where a and b are functions of t, what is du/dt?
13. Describe the collection of vectors tangent to all possible curves on the paraboloid $z = x^2 + y^2$ through the point $(1, 2, 5)$.
14. Show that if a surface is defined by an equation $f(x, y, z) = 0$, and if $(x(t), y(t), z(t))$ is a curve in the surface which passes through the point (x_0, y_0, z_0) when $t = t_0$, then the two vectors $x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}$ and $f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}$ are perpendicular.
15. (a) Use the chain rule to find $(d/dx)(x^x)$ by using the function $f(y, z) = y^z$.
    (b) Calculate $(d/dx)(x^x)$ by using one-variable calculus.
    (c) Which way do you prefer?
16. Suppose that the temperature at the point (x, y, z) in space is $T(x, y, z) = x^2 + y^2 + z^2$. Let a particle follow the right circular helix $\boldsymbol{\sigma}(t) = (\cos t, \sin t, t)$ and let $T(t)$ be its temperature at time t.
    (a) What is $T'(t)$?
    (b) Find an approximate value for the temperature at $t = (\pi/2) + 0.01$.
17. Use the chain rule to find a formula for the derivative $(d/dt)(f(t)g(t)/h(t))$.
18. Use the chain rule to differentiate $f(t)/[g(t)h(t)]$.

★19. A bug is swimming along the surface of a wave as in the Supplement to Section 15.1. Suppose that the motion of this wave is described by the function $f(t, y) = e^{-y}\cos t + \sin(y + t^2)$. At $t = 2$, the bug is at the position $y = 3$ and its horizontal velocity dy/dt is equal to 5. What is its vertical velocity dz/dt at that moment?

★20. Prove the chain rule by filling in the details in the following argument. Let $z = f(g(t), h(t))$, where g and h are differentiable and f has continuous partial derivatives.

(a) Show that

$$\frac{\Delta z}{\Delta t} = \frac{1}{\Delta t}\{[f(g(t + \Delta t), h(t + \Delta t)) - f(g(t), h(t + \Delta t)] + [f(g(t), h(t + \Delta t)) - f(g(t), h(t))]\}.$$

(b) Apply the mean value theorem for functions of one variable to each of the expressions in square brackets.

(c) Take the limit as $\Delta t \to 0$. (You will use the continuity of partial derivatives at this point.)

★21. Suppose that $z = f(x, y)$ is a surface with the property that if (x_0, y_0, z_0) lies on the surface, then so does the half-line from the origin through (x_0, y_0, z_0). Prove that this half-line also lies in the tangent plane to $z = f(x, y)$ at (x_0, y_0). Give an explicit example of such a surface.

★22. The differential equation $u_t + u_{xxx} + uu_x = 0$, called the *Korteweg–de Vries equation*, describes the motion of water waves in a shallow channel. Show that for any positive number c, the function

$$u(x, t) = 3c\,\text{sech}^2\left[\tfrac{1}{2}(x - ct)\sqrt{c}\,\right]$$

is a solution of the Korteweg–de Vries equation. This solution represents a travelling "hump" of water in the channel and is called a *soliton*. How do the shape and speed of the soliton depend on c? (Solitons were first discovered by J. Scott Russell around 1840 in barge canals near Edinburgh. He reported his results in the Transactions of the Royal Society of Edinburgh, 1840, Vol. **14**, pp. 47–109.)

15.4 Matrix Multiplication and the Chain Rule

The derivative matrix of a composite function is the product of two matrices.

The chain rule of Section 2.2 enabled us to differentiate a function which depended on one independent variable through one intermediate variable. In Section 15.3, this result was extended to the case of two or three intermediate variables. When we allow the number of functions and independent variables to grow to two or three, the chain rule may be expressed in terms of matrix multiplication.

An $m \times n$ *matrix* is a rectangular array of mn numbers, called the *entries* of the matrix, arranged in m rows and n columns. The entry in the ith row and jth column is called the (i, j) entry. (See Fig. 15.4.1.)

$$\begin{array}{r c} & \begin{array}{ccc} \text{First} & \text{Second} & \text{Third} \\ \text{column} & \text{column} & \text{column} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{r} \text{First row} \rightarrow \\ \text{Second row} \rightarrow \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \end{array}$$

Figure 15.4.1. The (i, j) entry of this matrix is a_{ij}.

The chain rule of Section 15.3 involves both the partial derivatives of one function of three variables and the derivatives of three functions of one variable. We may assemble the derivatives of one function of three variables as a 1×3 matrix or row vector which we denote

$$\left[\frac{\partial u}{\partial x} \; \frac{\partial u}{\partial y} \; \frac{\partial u}{\partial z} \right] = \frac{\partial u}{\partial(x, y, z)} \tag{1}$$

and the velocity vector of three functions of one variable as a 3×1 matrix or column vector which we denote

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \\ \dfrac{dz}{dt} \end{bmatrix} = \frac{\partial(x, y, z)}{\partial t}\,. \tag{2}$$

(If there are only two intermediate variables, our row and column vectors will be 1×2 and 2×1 matrices.)

To express the chain rule in this new notation, we define a product between row and column vectors of the same length.

Multiplication of Row and Column Vectors

Let

$$A = [a_1 a_2 \ldots a_n] \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

be a row vector and a column vector, respectively. If $m = n$, we define the product AB to be the number $a_1b_1 + a_2b_2 + \cdots a_nb_n = \sum_{i=1}^{n} a_i b_i$. (If $m \neq n$, the product AB is not defined.)

In terms of this definition, the chain rule for three intermediate variables becomes

$$\frac{du}{dt}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\end{bmatrix}\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\\ \frac{dz}{dt}\end{bmatrix}=\frac{\partial u}{\partial(x,y,z)}\,\frac{\partial(x,y,z)}{\partial t}, \tag{3}$$

which looks very much like the chain rule of one-variable calculus.

The product of row and column vectors has many other applications. For example, every linear function $f(x, y, z) = ax + by + cz + d$ can be written as

$$f(x,y,z)=\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}+d.$$

Your bill at the fruit market can be expressed as a product PQ, where

$$P=\begin{bmatrix}p_1 & p_2 & \cdots & p_n\end{bmatrix}$$

is the *price vector* whose ith entry is the price of the ith fruit in dollars per kilogram, and

$$Q=\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}$$

is the *quantity vector* whose ith entry is the number of kilograms of the ith fruit purchased.

Example 1 Find AB if

$$A=\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}-1\\ 1\\ -1\\ 1\end{bmatrix}.$$

Solution $AB = (1)(-1) + (2)(1) + (3)(-1) + (4)(1) = -1 + 2 - 3 + 4 = 2.$ ▲

Having described the derivative of m functions of one variable by an $m \times 1$ matrix, and the derivative of one function of n variables by a $1 \times n$ matrix, it is natural for us to describe the derivative of m functions of n variables by an $m \times n$ matrix. For example, if $x = f(u, v)$, $y = g(u, v)$, and $z = h(u, v)$, we may put all six partial derivatives into a 3×2 matrix:

$$\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{bmatrix}=\frac{\partial(x,y,z)}{\partial(u,v)}.$$

The rows of this matrix are the derivative vectors of f, g, and h. The columns are the "partial velocity vectors" with respect to u and v of the vector-valued function $\mathbf{r}(u, v) = (f(u, v), g(u, v), h(u, v))$.

In general, we may define the *derivative matrix* of m functions of n variables, as in the box on the following page.

Derivative Matrix

Let $u_1 = f_1(x_1, x_2, \ldots, x_n)$, $u_2 = f_2(x_1, x_2, \ldots, x_n), \ldots,$ and $u_m = f_m(x_1, x_2, \ldots, x_n)$ be m functions of the n variables $x_1, \ldots, x_n$. The *derivative matrix* of the u_i's with respect to the x_j's is the $m \times n$ matrix:

$$\frac{\partial(u_1, \ldots, u_m)}{\partial(x_1, \ldots, x_n)} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_m}{\partial x_1} & \dfrac{\partial u_m}{\partial x_2} & & \dfrac{\partial u_m}{\partial x_n} \end{bmatrix}$$

whose (i, j) entry is the partial derivative $\partial u_i / \partial x_j$.

The entries of the derivative matrix are functions of $(x_1, \ldots, x_n)$. If we fix values $(x_1^0, \ldots, x_n^0)$ for the independent variables, then the derivative matrix becomes a matrix of numbers and is denoted by

$$\left.\frac{\partial(u_1, \ldots, u_m)}{\partial(x_1, \ldots, x_n)}\right|_{(x_1^0, \ldots, x_n^0)}$$

Example 2 Let $u = x^2 + y^2$, $v = x^2 - y^2$, and $w = xy$. Find $\partial(u, v, w)/\partial(x, y)$ and evaluate

$$\left.\frac{\partial(u, v, w)}{\partial(x, y)}\right|_{(-2,3)}$$

Solution Applying the definition, with $m = 3$, $n = 2$, $u_1 = u$, $u_2 = v$, $u_3 = w$, $x_1 = x$, and $x_2 = y$, we get

$$\frac{\partial(u, v, w)}{\partial(x, y)} = \begin{bmatrix} 2x & 2y \\ 2x & -2y \\ y & x \end{bmatrix}.$$

Substituting $x = -2$ and $y = 3$, we get

$$\left.\frac{\partial(u, v, w)}{\partial(x, y)}\right|_{(-2,3)} = \begin{bmatrix} -4 & 6 \\ -4 & -6 \\ 3 & -2 \end{bmatrix}. \blacktriangle$$

Notice that the derivative matrix of one function $u = f(t)$ of one variable is a 1×1 matrix $\partial(u)/\partial(t)$ whose single entry is just the ordinary derivative du/dt. Thus the chain rule (3) can be rewritten as

$$\frac{\partial(u)}{\partial(t)} = \frac{\partial(u)}{\partial(x, y, z)} \frac{\partial(x, y, z)}{\partial(t)}. \tag{4}$$

In the remainder of this section, we will show how to multiply matrices of all sizes and, thereby, to generalize the chain rule (4) to several independent and dependent variables.

Example 3 (a) Suppose that $u = ax + by + cz + d$ and $v = ex + fy + gz + h$, where $a, b, \ldots, h$ are constants.

(i) Express u and v by using products of row and column vectors.
(ii) Find the derivative matrix $\partial(u,v)/\partial(x,y,z)$.

(b) Suppose that x, y, and z in (a) are linear functions of t:

$x = mt + n, y = pt + q$, *and* $z = rt + s$.

(i) Express u and v in terms of t and find the derivative matrix $\partial(u,v)/\partial(t)$.
(ii) Express the elements of $\partial(u,v)/\partial(t)$ as products of row and column vectors.

Solution (a) (i) We have $u = [a\ b\ c]\begin{bmatrix} x \\ y \\ z \end{bmatrix} + d$

and $v = [e\ f\ g]\begin{bmatrix} x \\ y \\ z \end{bmatrix} + h$.

(ii) The derivative matrix is

$$\frac{\partial(u,v)}{\partial(x,y,z)} = \begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix},$$

all of whose entries are constants.

(b) (i) Substituting for x, y, and z their expressions in t, we get

$$u = a(mt + n) + b(pt + q) + c(rt + s) + d,$$
$$v = e(mt + n) + f(pt + q) + g(rt + s) + h.$$

We can find the derivative matrix without multiplying out:

$$\frac{\partial(u,v)}{\partial(t)} = \begin{bmatrix} \dfrac{du}{dt} \\ \dfrac{dv}{dt} \end{bmatrix} = \begin{bmatrix} am + bp + cr \\ em + fp + gr \end{bmatrix}.$$

(ii) The entries of $\partial(u,v)/\partial(t)$ are

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} m \\ p \\ r \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e & f & g \end{bmatrix}\begin{bmatrix} m \\ p \\ r \end{bmatrix}.$$

Notice that they are obtained by multiplying the rows of $\partial(u,v)/\partial(x,y,z)$ by the (single) column of $\partial(x,y,z)/\partial(t)$. ▲

The preceding example and the multiplication of row and column vectors suggest how we should multiply $m \times n$ matrices.

Matrix Multiplication

Let A and B be two matrices and assume that the number of columns of A equals the number of rows of B. To form $C = AB$:

1. Take the product of the first row of A and first column of B and let it be the $(1,1)$ entry of C.
2. Take the product of the first row and second column of B and let it be the $(1,2)$ entry of C.
3. Repeat. In general, the product of the ith row of A and jth column of B is the (i,j) entry of C.

Example 4 Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}.$$

Find AB and BA.

Solution

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + (-1)\cdot 1 & \\ & \\ & \end{bmatrix} = \begin{bmatrix} 1 & \\ & \\ & \end{bmatrix}$$

(going across the first row of A and down the first column of B). Moving to the second column of B:

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1\cdot 0 + (-1)(-1) \\ & \\ & \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & \\ & \end{bmatrix}.$$

Moving to the second and third rows of A, we fill in the remaining entries:

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 0 \\ 13 & 3 \end{bmatrix}.$$

BA is not defined since the number of columns of B is not equal to the number of rows of A. ▲

Example 5 Find:

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}. ▲$$

Example 5 shows that even if AB and BA are defined, they may not be equal. In other words, matrix multiplication is *not commutative*.

Example 6 For 2×2 matrices A and B, verify that $|AB| = |A||B|$, where $|A|$ denotes the determinant of A (Section 13.6).

Solution Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

Then $|A| = ad - bc$, $|B| = eh - gf$, and

$$\begin{aligned} |AB| &= \begin{vmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{vmatrix} \\ &= (ae + bg)(cf + dh) - (ce + dg)(af + bh) \\ &= aecf + aedh + bgcf + bgdh - aecf - adgf - cebh - bgdh \\ &= aedh + bgcf - adgf - cebh \\ &= (ad - bc)(eh - gf) \\ &= |A| \cdot |B|. ▲ \end{aligned}$$

The result of Example 3(b) may be written in the following way in terms of derivative matrices:

$$\frac{\partial(u,v)}{\partial(t)} = \frac{\partial(u,v)}{\partial(x,y,z)}\,\frac{\partial(x,y,z)}{\partial(t)}\,.$$

This suggests a similar formula for the general chain rule.

The General Chain Rule

Let $u_1 = f_1(x_1, \ldots, x_n), \ldots,\ u_m = f_m(x_1, \ldots, x_n)$ be m functions of n variables, and let $x_1 = g_1(t_1, \ldots, t_k), \ldots,\ x_n = g_n(t_1, \ldots, t_k)$ be n functions of k variables, all with continuous partial derivatives.

Consider the u_i's as functions of the t_j's by

$$u_1 = f_1(g_1(t_1, \ldots, t_k), \ldots, g_n(t_1, \ldots, t_k)).$$

Then

$$\frac{\partial(u_1, \ldots, u_m)}{\partial(t_1, \ldots, t_k)} = \frac{\partial(u_1, \ldots, u_m)}{\partial(x_1, \ldots, x_n)}\,\frac{\partial(x_1, \ldots, x_n)}{\partial(t_1, \ldots, t_k)}\,.$$

In other words,

$$\frac{\partial u_i}{\partial t_j} = \frac{\partial u_i}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial u_i}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial u_i}{\partial x_n}\frac{\partial x_n}{\partial t_j}\,.$$

(Note that there are as many terms in the sum as there are intermediate variables.)

We will carry out the proof for the "typical" case $m=2$, $n=3$, $k=2$. We must prove that

$$\begin{bmatrix} \dfrac{\partial u_1}{\partial t_1} & \dfrac{\partial u_1}{\partial t_2} \\[2ex] \dfrac{\partial u_2}{\partial t_1} & \dfrac{\partial u_2}{\partial t_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\[2ex] \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_1}{\partial t_2} \\[2ex] \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_2}{\partial t_2} \\[2ex] \dfrac{\partial x_3}{\partial t_1} & \dfrac{\partial x_3}{\partial t_2} \end{bmatrix}.$$

This matrix equation represents four ordinary equations. We will prove a typical one:

$$\frac{\partial u_2}{\partial t_1} = \frac{\partial u_2}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial x_3}{\partial t_1}\,. \tag{5}$$

In taking the partial derivatives with respect to t_1, we hold t_2 fixed and take ordinary derivatives with respect to t_1. With this understood, we may rewrite (5) as

$$\frac{du_2}{dt_1} = \frac{\partial u_2}{\partial x_1}\frac{dx_1}{dt_1} + \frac{\partial u_2}{\partial x_2}\frac{dx_2}{dt_1} + \frac{\partial u_2}{\partial x_3}\frac{dx_3}{dt_1}\,. \tag{6}$$

We are now in the situation of Section 15.3—we have the independent variable t_1, the dependent variable u_2, and intermediate variables (x_1, x_2, x_3); but the chain rule for this case is just formula (6), so (6) is true and hence (5) is proved.

Example 7 Verify the chain rule for $\partial p/\partial x$, where

$$p = f(u,v,w) = u^2 + v^2 - w, \quad u = x^2y, \quad v = y^2, \quad \text{and} \quad w = e^{-xz}.$$

Solution $f(u,v,w) = (x^2y)^2 + y^4 - e^{-xz} = x^4y^2 + y^4 - e^{-xz}$. Thus

$$\frac{\partial p}{\partial x} = 4x^3y^2 + ze^{-xz}.$$

On the other hand,

$$\begin{aligned}\frac{\partial p}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial p}{\partial w}\frac{\partial w}{\partial x} &= 2u(2xy) + 2v \cdot 0 + ze^{-xz} \\ &= (2x^2y)(2xy) + ze^{-xz},\end{aligned}$$

which is the same. ▲

Example 8 Let (x, y) be cartesian coordinates in the plane and let (r,θ) be polar coordinates. (a) If $z = f(x, y)$ is a function on the plane, express the partial derivatives $\partial z/\partial r$ and $\partial z/\partial\theta$ in terms of $\partial z/\partial x$ and $\partial z/\partial y$. (b) Express $\partial^2 z/\partial r^2$ in cartesian coordinates.

Solution (a) By the general chain rule, using $x = r\cos\theta$ and $y = r\sin\theta$,

$$\begin{aligned}\begin{bmatrix} \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial\theta} \end{bmatrix} &= \begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.\end{aligned}$$

Multiplying out,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta,$$

$$\frac{\partial z}{\partial\theta} = r\left[-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\right].$$

(b) By (a),

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta. \tag{7}$$

Thus $\dfrac{\partial^2 z}{\partial r^2} = \dfrac{\partial}{\partial r}\left[\dfrac{\partial z}{\partial x}\right]\cos\theta + \dfrac{\partial}{\partial r}\left[\dfrac{\partial z}{\partial y}\right]\sin\theta$. Applying equation (7) with $\partial z/\partial x$ and $\partial z/\partial y$ replacing z, we get

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 z}{\partial x\,\partial y}\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin\theta\right)\sin\theta \\ &= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta \\ &= \frac{1}{x^2+y^2}\left[x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2}\right]. \text{ ▲}\end{aligned}$$

Example 9 Suppose that $(t,s) = (f(x, y), g(x, y))$, $x = u - 2v$, and $y = u + 3v$. Express the derivative matrix $\partial(t,s)/\partial(u,v)$ in terms of $\partial(t,s)/\partial(x, y)$.

Solution By the chain rule,

$$\frac{\partial(t,s)}{\partial(u,v)} = \frac{\partial(t,s)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}.$$

In this example,

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix},$$

so

$$\frac{\partial(t,s)}{\partial(u,v)}=\frac{\partial(t,s)}{\partial(x,y)}\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}=\begin{bmatrix}\dfrac{\partial t}{\partial x} & \dfrac{\partial t}{\partial y}\\[2mm] \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y}\end{bmatrix}\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{\partial t}{\partial x}+\dfrac{\partial t}{\partial y} & -2\dfrac{\partial t}{\partial x}+3\dfrac{\partial t}{\partial y}\\[2mm] \dfrac{\partial s}{\partial x}+\dfrac{\partial s}{\partial y} & -2\dfrac{\partial s}{\partial x}+3\dfrac{\partial s}{\partial y}\end{bmatrix}. \blacktriangle$$

Exercises for Section 15.4

Find the matrix products in Exercises 1–4.

1. $[1\ \ 2\ \ 3]\begin{bmatrix}4\\5\\6\end{bmatrix}$

2. $\begin{bmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}$

3. $[2\ \ 2\ \ 6]\begin{bmatrix}3\\4\\5\end{bmatrix}$

4. $[1\ \ 2\ \ 3\ \ 4\ \ 5]\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$

Find the derivative matrices in Exercises 5–8 and evaluate at the given points.

5. $\partial(x,y)/\partial(u,v)$; $x=u\sin v$, $y=e^{uv}$; at $(0,1)$.
6. $\partial(x,y,z)/\partial(r,\theta,\phi)$; where $x=r\sin\phi\cos\theta$, $y=r\sin\phi\sin\theta$, $z=r\cos\phi$; at $(2,\pi/3,\pi/4)$.
7. $\partial(u,v)/\partial(x,y,z)$; $u=xyz$, $v=x+y+z$; at $(3,3,3)$.
8. $\partial(x,y)/\partial(r,\theta)$; $x=r\cos\theta$, $y=r\sin\theta$; at $(5,\pi/6)$.

Compute the matrix products in Exercises 9–20 or explain why they are not defined.

9. $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$

10. $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

11. $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix}$

12. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix}$

13. $\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$

14. $\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix}$

15. $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$

16. $\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\3 & 2 & 1\end{bmatrix}$

17. $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix}$

18. $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix}$

19. $\left(\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}\begin{bmatrix}2 & 4\\1 & -1\end{bmatrix}\right)\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

20. $\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}\left(\begin{bmatrix}2 & 4\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\right)$

Compute $\partial z/\partial x$ and $\partial z/\partial y$ in Exercises 21–24 using matrix multiplication and by direct substitution.

21. $z=u^2+v^2$; $u=2x+7$, $v=3x+y+7$.
22. $z=u^2+3uv-v^2$; $u=\sin x$, $v=-\cos x+\cos y$.
23. $z=\sin u\cos v$; $u=3x^2-2y$, $v=x-3y$.
24. $z=u/v^2$; $u=x+y$, $v=xy$.
25. (a) Compute derivative matrices $\partial(x,y)/\partial(t,s)$ and $\partial(u,v)/\partial(x,y)$ if
$$x=t+s,\qquad y=t-s,$$
$$u=x^2+y^2,\qquad v=x^2-y^2.$$
(b) Express (u,v) in terms of (t,s) and calculate $\partial(u,v)/\partial(t,s)$.
(c) Verify that the chain rule holds.
26. Do as in Exercise 25 for the functions $x=t^2-s^2$, $y=ts$, $u=\sin(x+y)$, $v=\cos(x-y)$.
27. Do as in Exercise 25 for $x=ts$, $y=ts$; $u=x$, $v=-y$.
28. Do as in Exercise 25 for $x=t^2+s^2$, $y=t^2-s^2$, $z=2ts$; $u=xy$, $v=xz$, $w=xz$.
29. Suppose that a function is given in terms of rectangular coordinates by $u=f(x,y,z)$. If
$$x=r\cos\theta\sin\phi,$$
$$y=r\sin\theta\sin\phi,$$
$$z=r\cos\phi,$$
express $\partial u/\partial r$, $\partial u/\partial\theta$, and $\partial u/\partial\phi$ in terms of $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$.
30. Suppose that x,y,z are as in Exercise 29 and $u=x^2+y^2+z^2$. Find $\partial u/\partial r$, $\partial u/\partial\theta$, and $\partial u/\partial\phi$.
31. Express the polar coordinates r and θ in terms of the cartesian coordinates x and y, and find the derivative matrix $\partial(r,\theta)/\partial(x,y)$.

32. Let A be the derivative matrix $\partial(x, y)/\partial(r, \theta)$ for $x = r\cos\theta$, $y = r\sin\theta$. Let B be the derivative matrix $\partial(r, \theta)/\partial(x, y)$ of Exercise 31, with its entries expressed in terms of r and θ. Find AB and BA.

33. Let B be the $m \times 1$ column vector
$$\begin{bmatrix} \frac{1}{m} \\ \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{bmatrix}.$$
If $A = [a_1 \cdots a_m]$ is any row vector, what is AB?

Exercises 34–38 form a unit.

34. Let
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$
Find a matrix B such that $AB = I$.

35. In Exercise 34, show that we also have
$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

36. Show that the solution of the equation
$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} e \\ f \end{bmatrix},$$
where A and B are as in Exercises 34 and 35.

37. Find a matrix B such that
$$B\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

38. Using the results of Exercises 36 and 37, solve each of the following systems of equations:
(a) $x + 2y = 1$, $2x + 5y = 2$;
(b) $x + 2y = 0$, $2x + 5y = 0$.

Exercises 39–42 form a unit.

39. If $(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)) = (u_1, \ldots, u_n)$ are n functions of n variables, then the (square) matrix of partial derivatives is called the *jacobian matrix*. Its determinant is called the *jacobian determinant* and is denoted by
$$\left| \frac{\partial(u_1, \ldots, u_n)}{\partial(x_1, \ldots, x_n)} \right|.$$
(a) Suppose that $n = 2$. Show that the absolute value of
$$\left| \frac{\partial(u_1, u_2)}{\partial(x, y)} \bigg|_{(a,b)} \right|$$
is the area of the parallelogram spanned by
$$\left(\frac{\partial u_1}{\partial x}\bigg|_{(a,b)}, \frac{\partial u_2}{\partial x}\bigg|_{(a,b)} \right)$$
and
$$\left(\frac{\partial u_1}{\partial y}\bigg|_{(a,b)}, \frac{\partial u_2}{\partial y}\bigg|_{(a,b)} \right).$$
(b) Suppose that $n = 3$. Show that the absolute value of
$$\left| \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} \bigg|_{(a,b,c)} \right|$$
is the volume of the parallelepiped spanned by the vectors
$$\left(\frac{\partial u_1}{\partial x_i}\bigg|_{(a,b,c)}, \frac{\partial u_2}{\partial x_i}\bigg|_{(a,b,c)}, \frac{\partial u_3}{\partial x_i}\bigg|_{(a,b,c)} \right)$$
for $i = 1, 2, 3$.

40. Compute the following jacobian determinants:
(a) $(x, y) = (r\cos\theta, r\sin\theta)$. Find
$$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right|.$$
(b) Let $(x, y, z) = (r\cos\theta, r\sin\theta, z)$. Find
$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right|.$$
(c) Let $(x, y, z) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$. Find
$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \right|.$$

41. Compute the jacobian determinants (see Exercise 39) of the following functions at the indicated points:
(a) $(x, y) = (t^2 + s^2, t^2 - s^2)$; $(t, s) = (1, 2)$.
(b) $(u, v) = (x + y, xy)$; $(x, y) = (5, -3)$.
(c) Compute the jacobian determinant of (u, v) with respect to (t, s) from parts (a) and (b) at $(t, s) = (1, 2)$. Verify that your answer is the product of the answers in (a) and (b).

42. Prove the following equations (notation from Exercise 39) in light of the chain rule and the multiplicative property of determinants found in Example 6:
(a) $$\left| \frac{\partial(u, v)}{\partial(x, y)} \right| \left| \frac{\partial(x, y)}{\partial(t, s)} \right| = \left| \frac{\partial(u, v)}{\partial(t, s)} \right|;$$
(b) $$\left| \frac{\partial(x, y)}{\partial(t, s)} \right| \left| \frac{\partial(t, s)}{\partial(x, y)} \right| = 1.$$

43. Let v_1, v_2, v_3 be the components of a vector function $\mathbf{v}$, u a scalar function, a, b, ρ constants. Express in matrix notation the equations of elasticity:
$$\rho\left(\frac{\partial^2 v_1}{\partial t^2} \right) = (a + b)\left(\frac{\partial u}{\partial x} \right) + b\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right);$$
$$\rho\left(\frac{\partial^2 v_2}{\partial t^2} \right) = (a + b)\left(\frac{\partial u}{\partial y} \right) + b\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2} \right);$$
$$\rho\left(\frac{\partial^2 v_3}{\partial t^2} \right) = (a + b)\left(\frac{\partial u}{\partial z} \right) + b\left(\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial y^2} + \frac{\partial^2 v_3}{\partial z^2} \right).$$

44. A rotation of points in the xy plane (relative to fixed axes) is given by
$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
where θ is the angle of rotation of
$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} X \\ Y \end{bmatrix}$$
Show by means of matrix multiplication that a rotation of θ_1 followed by a rotation of θ_2 is the same as a rotation of θ_2 followed by a rotation of θ_1.

45. The coordinates u, φ, θ are defined by $x = au \sin\varphi \cos\theta$, $y = bu \sin\varphi \sin\theta$, $z = cu \cos\varphi$ for $u \geqslant 0$, $0 \leqslant \varphi \leqslant \pi$, $0 \leqslant \theta \leqslant 2\pi$.
   (a) Show that the surfaces $u =$ constant are the *ellipsoids*
   $$\left(\frac{x}{au}\right)^2 + \left(\frac{y}{bu}\right)^2 + \left(\frac{z}{cu}\right)^2 = 1.$$
   (b) Show that the surfaces $\varphi =$ constant are *elliptical cones*.
   (c) Show the surface $\theta =$ constant is a *plane*.
   (d) Volume calculations involve the determinant of $\partial(x, y, z)/\partial(u, \varphi, \theta)$. Show that it equals $abcu^2 \sin\varphi$.

46. The matrix equation
$$\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
can be viewed as a *rotation* in the xy plane through the angle θ. (See Exercise 4.4). Similarly, the equation
$$\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
can be viewed as a *translation* in the xy plane.
   (a) Use a matrix multiplication to find the matrix equation for a rotation followed by a translation.
   (b) Is a rotation followed by a translation the same as a translation followed by a rotation?

★47. Verify the formula $|AB| = |A||B|$ for 3×3 matrices A and B, where $|A|$ denotes the determinant of A.

★48. Public health officials have located four persons, x_1, x_2, x_3, and x_4, known to be carrying a new strain of flu. Three persons, y_1, y_2, and y_3, report possible contact, and a first-order contact matrix A is set up whose i, jth entry is 1 if there was contact between x_i and y_j and zero otherwise. Five other people, z_1, z_2, z_3, z_4, and z_5, are questioned for possible contact with y_1, y_2, and y_3, and another first-order contact matrix B is set up whose i, jth entry is 1 if y_i has contacted z_j and zero otherwise.
   (a) Show that the product matrix $C = AB$ counts the number of second-order contacts. That is, the i, jth entry of C is the number of possible paths of disease communication from x_i to z_j.
   (b) Write down the three matrices for the situation shown in Fig. 15.4.2. Check the conclusion of part (a).

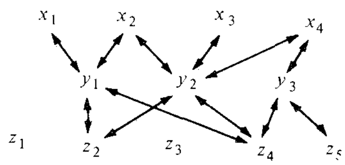

Figure 15.4.2. Contacts between three groups of people.

★49. Express Simpson's rule (Section 11.5) by using a product of row and column vectors.

★50. Suppose that f is a differentiable function of one variable and that a function $u = g(x, y)$ is defined by
$$u = g(x, y) = xyf\left(\frac{x+y}{xy}\right).$$
Show that u satisfies a (partial) differential equation of the form
$$x^2 \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} = G(x, y)u$$
and find the function $G(x, y)$.

Review Exercises for Chapter 15

Calculate all first partial derivatives for the functions in Exercises 1–10.

1. $u = g(x, y) = \dfrac{\sin(\pi x)}{1 + y^2}$.
2. $u = f(x, z) = \dfrac{x}{1 + \cos(2z)}$.
3. $u = k(x, z) = xz^2 - \cos(xz^3)$.
4. $u = m(y, z) = y^z$.
5. $u = h(x, y, z) = zx + y^2 + yz$.
6. $u = n(x, y, z) = x^{yz}$.
7. $u = f(x, y, z) = \ln[1 + e^{-x}\cos(xy)]$.
8. $u = h(x, y, z) = \cos(e^{-x^2 - 2y^2})$.
9. $u = g(x, y, z) = xz + e^z\left(\int_0^x t^2 e^t \, dt\right)$.
10. $u = f(x, y) = \cos(xy^2) + \exp\left[\int_0^x \sqrt{t} \cos(ty) \, dt\right]$.

Check the equality of the given mixed partials for the indicated functions in Exercises 11–16.

11. $\partial^2 u/\partial x \, \partial y = \partial^2 u/\partial y \, \partial x$ for u in Exercise 1.
12. $\partial^2 u/\partial x \, \partial z = \partial^2 u/\partial z \, \partial x$ for u in Exercise 2.
13. $\partial^2 u/\partial x \, \partial z = \partial^2 u/\partial z \, \partial x$ for u in Exercise 3.
14. $\partial^2 u/\partial y \, \partial z = \partial^2 u/\partial z \, \partial y$ for u in Exercise 4.
15. $\partial^2 u/\partial x \, \partial z = \partial^2 u/\partial z \, \partial x$ for u in Exercise 5.
16. $\partial^2 u/\partial x \, \partial z = \partial^2 u/\partial z \, \partial x$ for u in Exercise 6.

17. Find $(\partial/\partial x)e^{x-\cos(yx)}|_{x=1,y=0}$.
18. Find $(\partial/\partial s)\exp(rs^3 - r^3s)|_{r=1,s=1}$.
19. Find $f_x(1,0)$ if $f(x, y) = \cos(x + e^{yx})$.
20. Find $f_s(-1,2)$ if $f(r,s) = (r + s^2)/(1 - r^2 - s^2)$.
21. The possible time T in minutes of a scuba dive is given by $T = 32V/(x + 32)$, where V is the volume of air in cubic feet at 15 psi (pounds per square inch) which is compressed into the air tanks, and x is the depth of the scuba dive in feet.
    (a) How long can a 27-foot dive last when $V = 65$?
    (b) Find $\partial T/\partial x$ and $\partial T/\partial V$ when $x = 27$, $V = 65$. Interpret.
22. The displacement of a certain violin string placed on the x axis is given by $u = \sin(x - 6t) + \sin(x + 6t)$. Calculate the velocity of the string at $x = 1$ when $t = \frac{1}{3}$.

Find the limits in Exercises 23–26, if they exist.

23. $\lim_{(x,y)\to(0,0)} (x^2 - 2xy + 4)$
24. $\lim_{(x,y)\to(0,0)} (x^3 - y^3 + 15)$
25. $\lim_{(x,y)\to(0,0)} \dfrac{x^3 - y^3}{x^2 + y^2}$
26. $\lim_{(x,y)\to(0,0)} \dfrac{xy + x^3 + x - 2}{\sqrt{x^2 + y^2}}$

Find the equation of the tangent plane to the given surface at the indicated point in Exercises 27–30.

27. $z = x^2 + y^2$; $x = 1, y = 1$.
28. $z = x \sin y$; $x = 2, y = \pi/4$.
29. $z = e^{xy}$; $x = 0, y = 0$.
30. $z = \sqrt{x^2 + y^2}$; $x = 3, y = 4$.

Use the linear approximation to find approximations for the quantities in Exercises 31–34.

31. $\sqrt{(1.01)^2 + (4.01)^2 + (8.002)^2}$
32. $(2.004)\ln(0.98)$
33. $(0.999)^{1.001}$
34. $(1.001)^{0.999}$
35. Find an approximate value for the hypotenuse of a right triangle whose legs are 3.98 and 3.03.
36. The capacitance per unit length of a parallel pair of wires of radii R and axis-to-axis separation D is given by
$$C = \frac{\pi\epsilon_0}{\ln\left(\dfrac{D + \sqrt{D^2 - 4R^2}}{2R}\right)}.$$
The capacitance between a wire and a plane parallel to it is
$$C^* = \frac{2\pi\epsilon_0}{\ln\left(\dfrac{h + \sqrt{h^2 - R^2}}{R}\right)},$$
where h = distance from the wire to the plane.
    (a) Find the expected change in capacitance for two parallel wires, separated by 2 centimeters, with radius 0.40 centimeter, due to a radius increase of 0.01 centimeter.
    (b) A wire of 0.57 centimeter radius has its central axis at a uniform distance of 3 centimeters from a conducting plane. Due to heating, the wire increases 0.02 centimeter in radius, but due to bowing of the wire, it can be assumed that the axis of the wire was raised to 3.15 centimeters above the plane. What is the expected change in capacitance?
37. At time $t = 0$, a particle is ejected from the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$ in a direction normal to the surface at a speed of 10 units per second. At what time does it cross the sphere $x^2 + y^2 + z^2 = 103$? [Hint: Solve for z].
38. At what point(s) on the surface in Exercise 37 is the normal vector parallel to the line $x = y = z$?
39. Verify the chain rule for the function $f(x, y, z) = \ln(1 + x^2 + 2z^2)/(1 + y^2)$ and the curve $\sigma(t) = (t, 1 - t^2, \cos t)$.
40. Verify the chain rule for the function $f(x, y) = x^2/(2 + \cos y)$ and the curve $x = e^t$, $y = e^{-t}$.
41. (a) Let c be a constant. Show that, for every function $f(x)$, the function $u(x,t) = f(x - ct)$ satisfies the partial differential equation $u_t + cu_x = 0$.
    (b) With u as in (a), consider for each value of t the graph $z = u(x,t)$ in the xz plane. How does this change as t increases?
42. (a) Show that, if $u(x,t)$ is *any* solution of the equation $u_t + cu_x = 0$, then the function $g(y,t)$ defined by $g(y,t) = u(y + ct, t)$ is independent of t.
    (b) Conclude from (a) that u must be of the form $u(x,t) = f(x - ct)$ for some function f.
    (c) What kind of wave motion is described by the equation $u_t + cu_x = 0$?
43. A right circular cone of sand is gradually collapsing. At a certain moment, the cone has a height of 10 meters and a base radius of 3 meters. If the height of the cone is decreasing at a rate of 1 meter per hour, how is the radius changing, assuming that the volume remains constant?
44. A boat is sailing northeast at 20 kilometers per hour. Assuming that the temperature drops at a rate of 0.2°C per kilometer in the northerly direction and 0.3°C per kilometer in the easterly direction, what is the time rate of change of temperature as observed on the boat?
45. Use the chain rule to find a formula for $(d/dt)\exp[f(t)g(t)]$.
46. Use the chain rule to find a formula for $(d/dt)(f(t)^{g(t)})$.

47. If x and y are functions of t,
$$\left.\frac{dx}{dt}\right|_{t=0}=1, \text{ and } \left.\frac{dy}{dt}\right|_{t=0}=-1,$$
find $\left.\frac{d}{dt}e^{x+2xy}\right|_{t=0}$ in terms of x and y.

48. If x, y, and z are functions of t and
$$\left.\frac{dx}{dt}\right|_{t=0}=1, \qquad \left.\frac{dy}{dt}\right|_{t=0}=0$$
and $\left.\frac{dz}{dt}\right|_{t=0}=-1$, find $\left.\frac{d}{dt}\cos(xyz^2)\right|_{t=0}$ in terms of x, y, and z.

49. The tangent plane to $z=x^2+6y^2$ at $x=1$, $y=1$ meets the xy plane in a line. Find the equation of this line.

50. The tangent plane to $z=e^{x-y}$ at $x=1$, $y=2$ meets the line $x=t$, $y=2t-1$, $z=5t$ in a point. Find it.

Find the products AB of the matrices in Exercises 51–60.

51. $A=[1\ \ 2\ \ 4]$, $B=\begin{bmatrix}2\\-1\\1\end{bmatrix}$

52. $A=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}$, $B=\begin{bmatrix}3/2\\3/2\\2\\2\end{bmatrix}$

53. $A=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, $B=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

54. $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$, $B=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$

55. $A=\begin{bmatrix}1&2\\2&4\end{bmatrix}$, $B=\begin{bmatrix}2&-1\\2&-1\end{bmatrix}$

56. $A=\begin{bmatrix}2&-1\\2&-1\end{bmatrix}$, $B=\begin{bmatrix}1&2\\2&4\end{bmatrix}$

57. $A=\begin{bmatrix}1&2\\-1&1\\2&1\end{bmatrix}$, $B=\begin{bmatrix}2&1&2\\-1&-1&1\end{bmatrix}$

58. $A=\begin{bmatrix}1&2&3&4&5\\6&7&8&9&10\end{bmatrix}$, $B=\begin{bmatrix}1\\-1\\1\\-1\\1\end{bmatrix}$

59. $A=\begin{bmatrix}3&0&1\\1&2&-1\\1&0&1\end{bmatrix}$, $B=\begin{bmatrix}1&0&-1\\2&0&1\\0&1&0\end{bmatrix}$

60. $A=\begin{bmatrix}1&0&-1\\2&0&1\\0&1&0\end{bmatrix}$, $B=\begin{bmatrix}3&0&1\\1&2&-1\\1&0&1\end{bmatrix}$

61. Compute $\partial z/\partial x$ and $\partial z/\partial y$ if
$$z=\frac{u^2+v^2}{u^2-v^2}, \qquad u=e^{-x-y}, \quad v=e^{xy}$$
by (a) substitution and (b) the chain rule.

62. Do as in Exercise 61 if $z=uv$, $u=x+y$, and $v=x-y$.

63. Suppose that $z=f(x,y)$, $x=u+v$ and $y=u-v$. Express $\partial z/\partial u$ and $\partial z/\partial v$ in terms of $\partial z/\partial x$ and $\partial z/\partial y$.

64. In the situation of Exercise 63, express $\partial z/\partial x$ and $\partial z/\partial y$ in terms of $\partial z/\partial u$ and $\partial z/\partial v$.

65. The *ideal gas law* $PV=nRT$ involves a constant R, the number n of moles of the gas, the volume V, the Kelvin temperature T, and the pressure P.
   (a) Show that each of n, P, T, V is a function of the remaining variables, and determine explicitly the defining equations.
   (b) The quantity $\partial P/\partial T$ is a rate of change. Discuss this in detail, and illustrate with an example which identifies the variables held constant.
   (c) Calculate $\partial V/\partial T, \partial T/\partial P, \partial P/\partial V$ and show that their product equals -1.

66. The *potential temperature* θ is defined in terms of temperature T and pressure p by
$$\theta=T\left(\frac{1000}{p}\right)^{0.286}.$$
The temperature and pressure may be thought of as functions of position (x,y,z) in the atmosphere and also of time t.
   (a) Find formulas for $\partial\theta/\partial x, \partial\theta/\partial y, \partial\theta/\partial z, \partial\theta/\partial t$ in terms of partial derivatives of T and p.
   (b) The condition $\partial\theta/\partial z<0$ is regarded as *unstable atmosphere*, for it leads to large vertical excursions of air parcels from a single upward or downward impetus. Meteorologists use the formula
$$\frac{\partial\theta}{\partial z}=\frac{\theta}{T}\left(\frac{\partial T}{\partial z}+\frac{g}{C_p}\right),$$
where $g=32.2$. $C_p=\text{constant}>0$. How does the temperature change in the upward direction for an unstable atmosphere?

67. The specific volume V, pressure P, and temperature T of a Van der Waals gas are related by $P=[RT/(V-\beta)]-\alpha/V^2$, where α, β, R are considered to be constants.
   (a) Explain why any two of V, P, or T can be considered independent variables which determine the third variable.
   (b) Find $\partial T/\partial P, \partial P/\partial V, \partial V/\partial T$. Identify which variables are constant, and interpret each partial derivative physically.
   (c) Verify that $(\partial T/\partial P)(\partial P/\partial V)(\partial V/\partial T)=-1$ (not $+1$!).

68. Dieterici's equation of state for a gas is
$$P(V-b)e^{a/RVT}=RT,$$
where a, b, and R are constants. Regard volume V as a function of temperature T and pressure P and show that
$$\frac{\partial V}{\partial T}=\left(R+\frac{a}{TV}\right)\left(\frac{RT}{V-b}-\frac{a}{V^2}\right)^{-1}.$$

69. What is wrong with the following argument? Suppose that $w = f(x, y)$ and $y = x^2$. By the chain rule,
$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial w}{\partial x} + 2x\frac{\partial w}{\partial y}.$$
Hence $0 = 2x(\partial w/\partial y)$, so $\partial w/\partial y = 0$.

70. What is wrong with the following argument? Suppose that $w = f(x, y, z)$ and $z = g(x, y)$. Then by the chain rule,
$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}.$$
Hence
$$0 = \frac{\partial w}{\partial z}\frac{\partial z}{\partial x},$$
so $\partial w/\partial z = 0$ or $\partial z/\partial x = 0$, which is, in general, absurd.

71. For a function u of three variables (x, y, z), show that $\partial^3 u/\partial x\,\partial y\,\partial z = \partial^3 u/\partial y\,\partial z\,\partial x$.

72. For a function u of three variables (x, y, z), show that $\partial^3 u/\partial x\,\partial y\,\partial z = \partial^3 u/\partial z\,\partial x\,\partial y$.

73. Prove that the functions
(a) $f(x, y) = \ln(x^2 + y^2)$,
(b) $g(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}$,
(c) $h(x, y, z, w) = \dfrac{1}{x^2 + y^2 + z^2 + w^2}$,
satisfy the respective Laplace equations:
(a) $f_{xx} + f_{yy} = 0$,
(b) $g_{xx} + g_{yy} + g_{zz} = 0$,
(c) $h_{xx} + h_{yy} + h_{zz} + h_{ww} = 0$,
where $f_{xx} = \partial^2 f/\partial x^2$, etc.

74. If $z = f(x - y)/y$, show that
$$z + y(\partial z/\partial x) + y(\partial z/\partial y) = 0.$$

75. Given $w = f(x, y)$ with $x = u + v$, $y = u - v$, show that
$$\frac{\partial^2 w}{\partial u\,\partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

★76. (a) A function $u = f(x_1, \ldots, x_m)$ is called *homogeneous of degree n* if
$$f(tx_1, \ldots, tx_m) = t^n f(x_1, \ldots, x_m).$$
Show that such a function satisfies *Euler's differential equation*
$$x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + \cdots + x_m\frac{\partial u}{\partial x_m} = nf(x_1, \ldots, x_m).$$
(b) Show that each of the following functions satisfies a differential equation of the type in part (a), find n, and check directly that f is homogeneous of degree n.
(i) $f(x, y) = x^2 + xy + y^2$;
(ii) $f(x, y, z) = x + 3y - \sqrt{xz}$; $xz > 0$;
(iii) $f(x, y, z) = xyz + x^3 - x^2y$.

★77. In Exercise 77 on page 775 we saw that the mixed partial derivatives of
$$z = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$
at $(0, 0)$ are not equal. Is this consistent with the graph in Fig. 15.R.1?

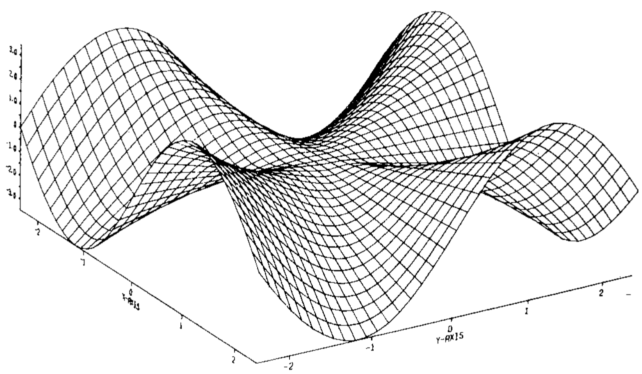

Figure 15.R.1. Computer-generated graph of
$$z = \frac{xy(x^2 - y^2)}{x^2 + y^2}.$$